CS 247 – Scientific Visualization
Lecture 9: Scalar Field Visualization, Pt. 3

Markus Hadwiger, KAUST

Reading Assignment #5 (until Mar 1)

Read (required):

- Gradients of scalar-valued functions

  `https://en.wikipedia.org/wiki/Gradient`

- Critical points

  `https://en.wikipedia.org/wiki/Critical_point_(mathematics)`

- Multivariable derivatives and differentials

  `https://en.wikipedia.org/wiki/Total_derivative`

  `https://en.wikipedia.org/wiki/Differential_of_a_function#`
  `                        Differentials_in_several_variables`

  `https://en.wikipedia.org/wiki/Hessian_matrix`

- Dot product, inner product (more general)

  `https://en.wikipedia.org/wiki/Dot_product`

  `https://en.wikipedia.org/wiki/Inner_product_space`

Basic contouring algorithms:

- cell-by-cell algorithms: simple structure, but generate disconnected segments, require post-processing

- contour propagation methods: more complicated, but generate connected contours

"Marching squares" algorithm (systematic cell-by-cell):

- process nodes in ccw order, denoted here as x_0, x_1, x_2, x_3

- compute at each node $\mathbf{x}_i$ the reduced field $\tilde{f}(x_i) = f(x_i) - (c - \varepsilon)$ (which is forced to be nonzero)

- take its sign as the ith bit of a 4-bit integer

- use this as an index for lookup table containing the connectivity information:

Contours in a quadrangle cell
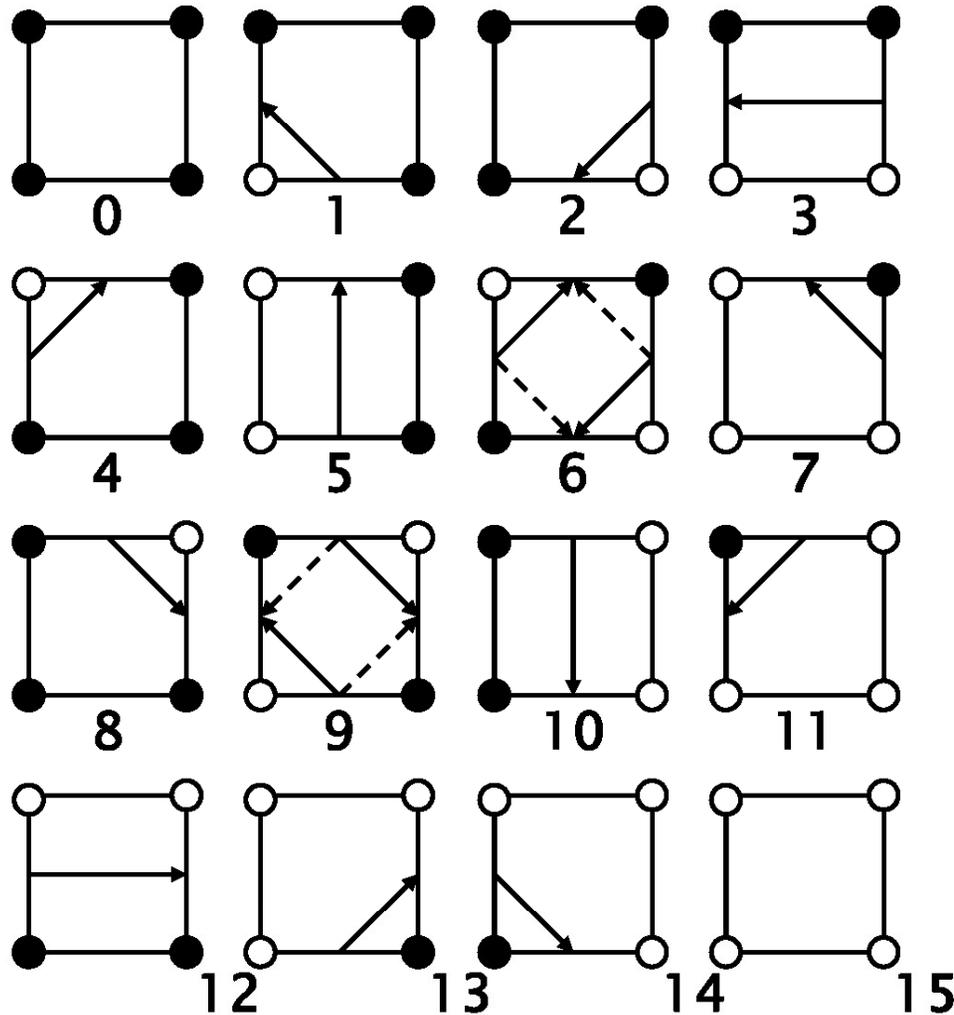
$$\bullet \quad \tilde{f}(x_i) < 0$$
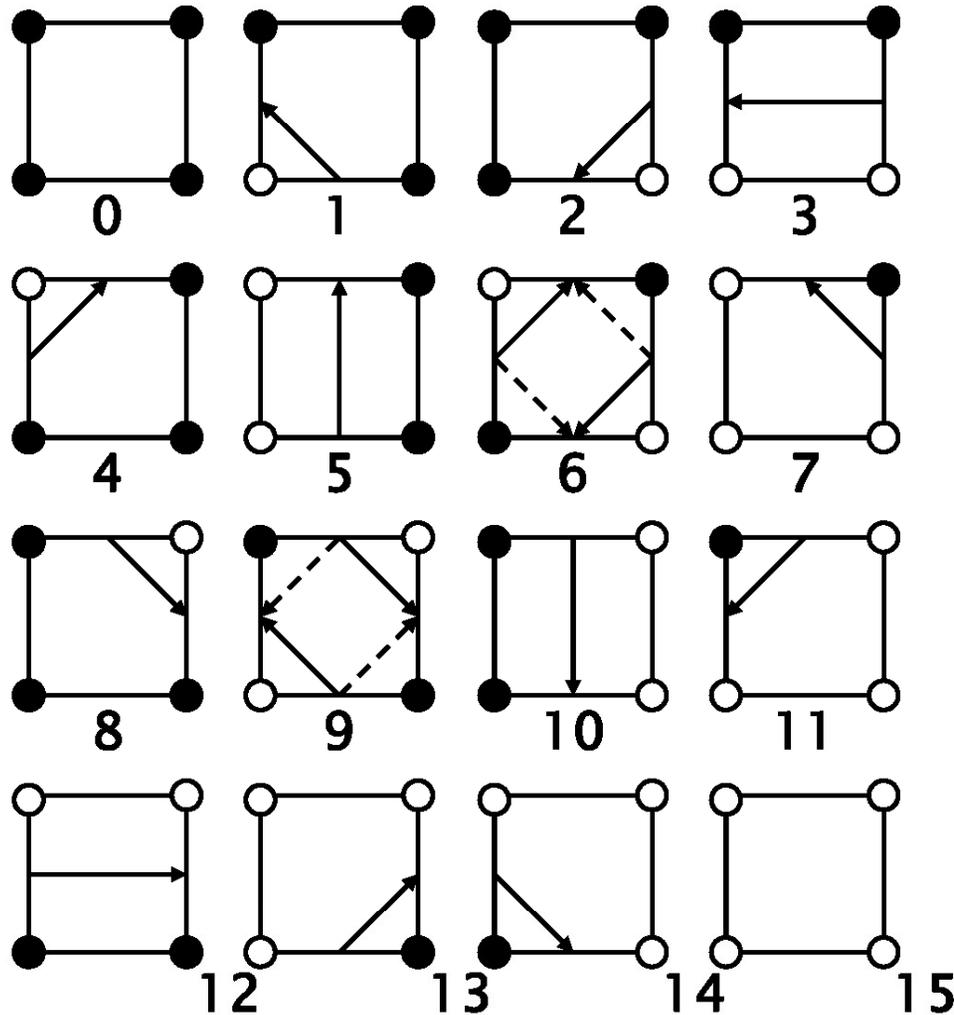
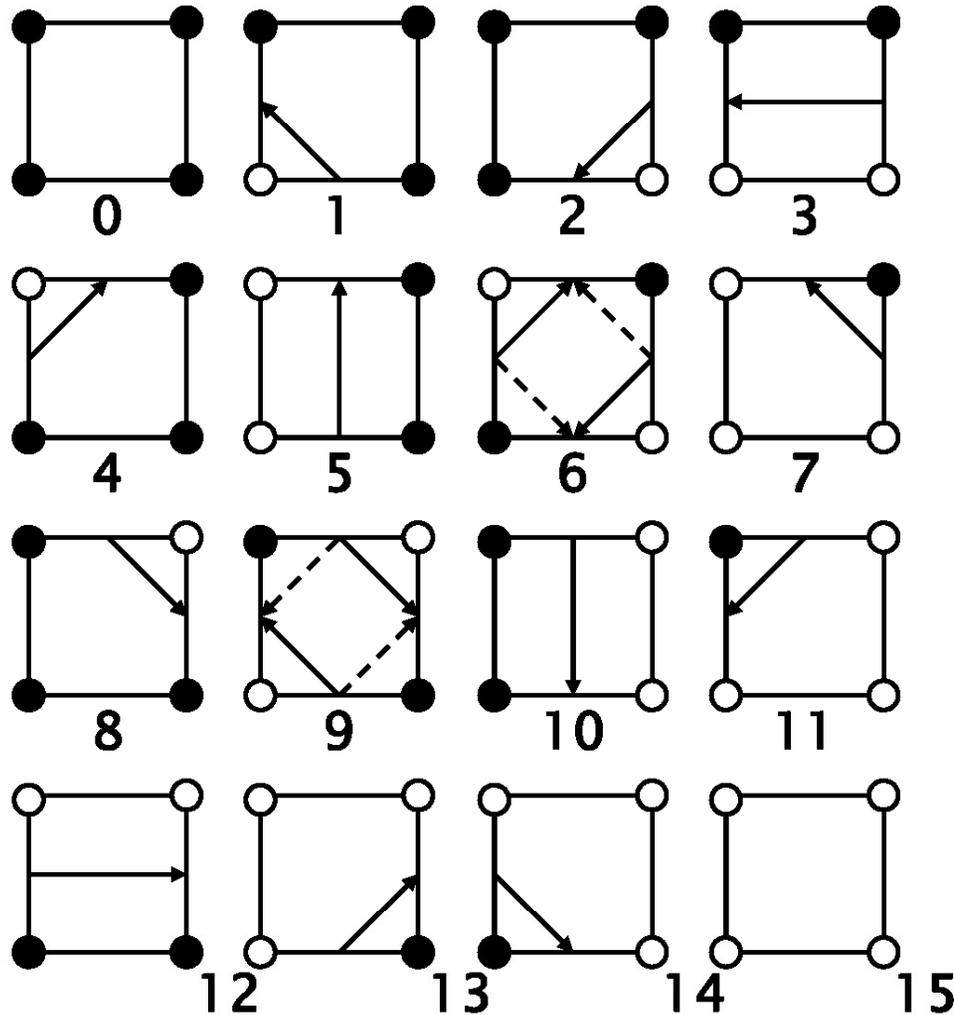$$\circ \quad \tilde{f}(x_i) > 0$$

Alternating signs exist in cases 6 and 9.

Choose the solid or dashed line?

Both are possible for topological consistency.

This allows to have a fixed table of 16 cases.

Contours in a quadrangle cell

$$\bullet \quad f(x_i) < c$$
$$\circ \quad f(x_i) \geq c$$

Alternating signs exist in cases 6 and 9.

Choose the solid or dashed line?

Both are possible for topological consistency.

This allows to have a fixed table of 16 cases.

Contours in a quadrangle cell

$$f(x_i) \leq c$$
$$f(x_i) > c$$

Alternating signs exist in cases 6 and 9.

Choose the solid or dashed line?

Both are possible for topological consistency.

This allows to have a fixed table of 16 cases.
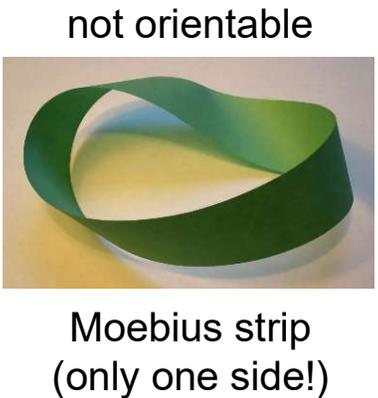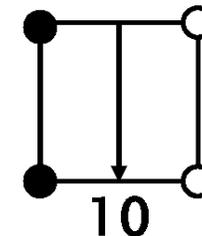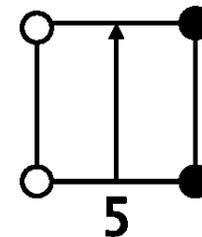
Orientability (1-manifold embedded in 2D)

Orientability of 1-manifold:

Possible to assign consistent left/right orientation

not orientable

Moebius strip
(only one side!)

Iso-contours

- Consistent side for scalar values…

  - greater than iso-value (e.g, *left* side)

  - less than iso-value (e.g., *right* side)

- Use consistent ordering of vertices
  (e.g., larger vertex index is "tip" of arrow;
  if (0,1) points "up", "left" is left, …)

$$\bullet \ \tilde{f}(x_i) < 0$$

$$\circ \ \tilde{f}(x_i) > 0$$

Orientability (2-manifold embedded in 3D)

Orientability of 2-manifold:

Possible to assign consistent normal vector orientation

not orientable

Moebius strip
(only one side!)

Triangle meshes

- Edges

  - Consistent ordering of vertices: CCW (counter-clockwise) or CW (clockwise)
    (e.g., (3,1,2) on one side of edge, (1,3,4) on the other side)

- Triangles

  - Consistent front side vs. back side

  - Normal vector; or ordering of vertices (CCW/CW)

  - See also: "right-hand rule"

2

3

1

`GL_CCW`

4
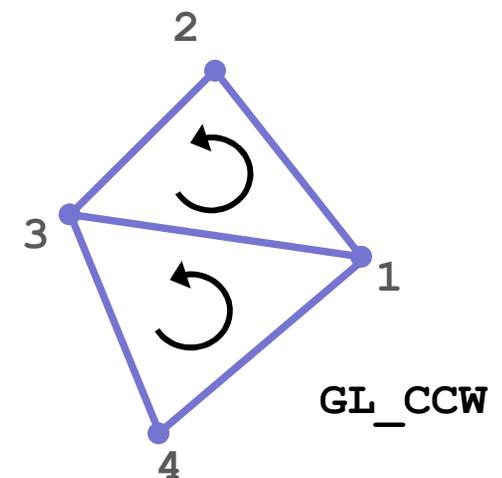
Topological consistency

To avoid degeneracies, use symbolic perturbations:

If level c is found as a node value, set the level to $c-\varepsilon$ where ε is a symbolic infinitesimal.

Then:

- contours intersect edges at some (possibly infinitesimal) distance from end points

- flat regions can be visualized by pair of contours at $c-\varepsilon$ and $c+\varepsilon$

- contours are topologically consistent, meaning:

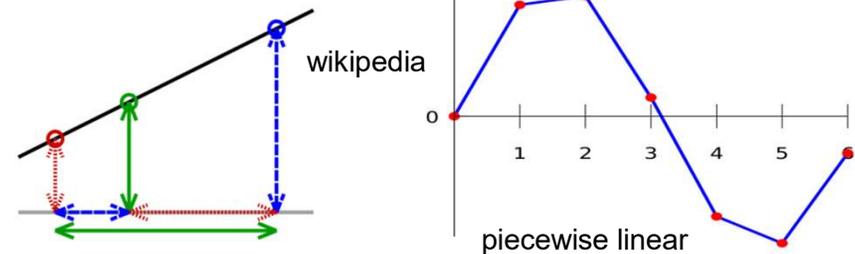Contours are closed, orientable, nonintersecting lines.
(except where the boundary is hit)

Linear Interpolation / Convex Combinations

Linear interpolation in 1D:

$$f(\alpha) = (1 - \alpha)v_1 + \alpha v_2$$

wikipedia

piecewise linear

Line embedded in 2D (linear interpolation of vertex coordinates/attributes):

$$f(\alpha_1, \alpha_2) = \alpha_1 v_1 + \alpha_2 v_2$$
$$\alpha_1 + \alpha_2 = 1$$

$$f(\alpha) = v_1 + \alpha(v_2 - v_1)$$
$$\alpha = \alpha_2$$

Line segment: $\qquad \alpha_1, \alpha_2 \geq 0 \qquad (\rightarrow$ convex combination)

Compare to line parameterization
with parameter t:

$$v(t) = v_1 + t(v_2 - v_1)$$

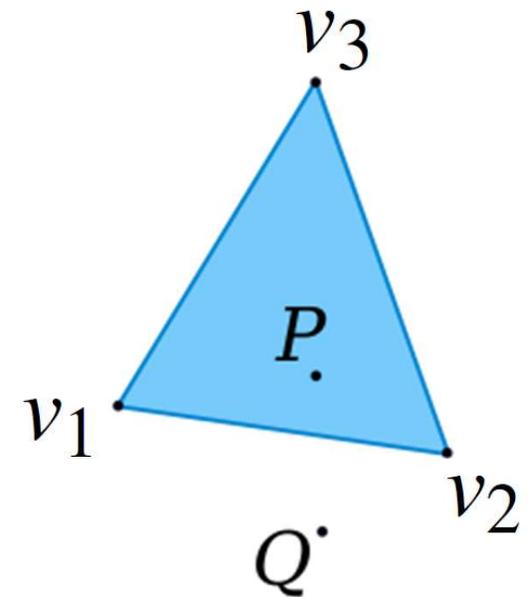**Linear** combination (n-dim. space):

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$$

Affine combination: Restrict to $(n-1)$-dim. subspace:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \sum_{i=1}^{n} \alpha_i = 1$$

Convex combination:

$$\alpha_i \geq 0$$

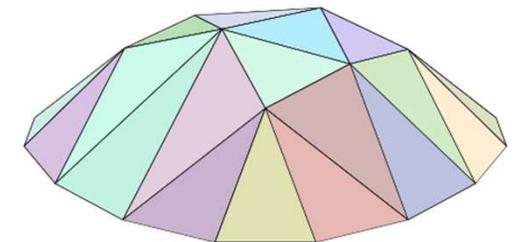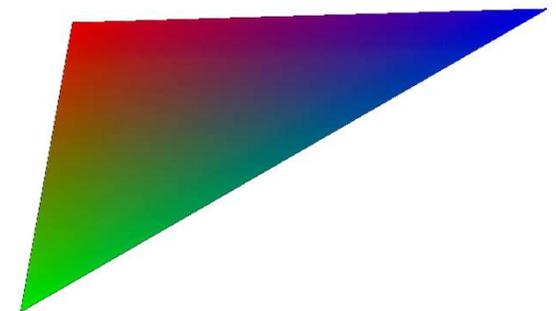(restrict to simplex in subspace)

Linear Interpolation / Convex Combinations

The weights α_i are the n normalized ***barycentric*** coordinates

$\rightarrow$ linear attribute interpolation in simplex

attribute interpolation

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$$
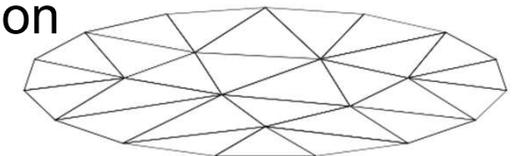
$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \sum_{i=1}^{n} \alpha_i = 1$$

$$\alpha_i \geq 0$$

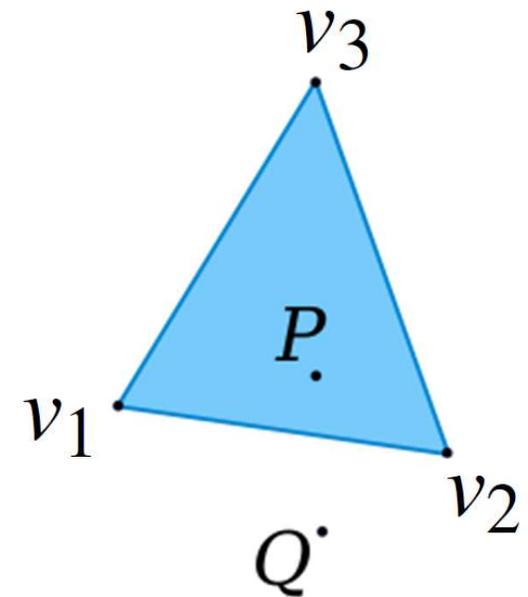spatial position
interpolation

wikipedia

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \sum_{i=1}^{n} \alpha_i = 1$$
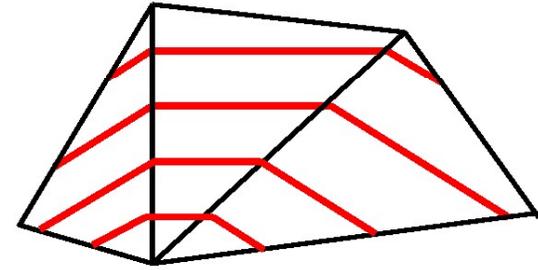
Can re-parameterize to get $(n-1)$ **affine** coordinates:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 =$$
$$\tilde{\alpha}_1 (v_2 - v_1) + \tilde{\alpha}_2 (v_3 - v_1) + v_1$$
$$\tilde{\alpha}_1 = \alpha_2$$
$$\tilde{\alpha}_2 = \alpha_3$$

Contours in triangle/tetrahedral cells

Linear interpolation of cells implies
   piece-wise linear contours.

Contours are unambiguous, making
   "marching triangles" even simpler than
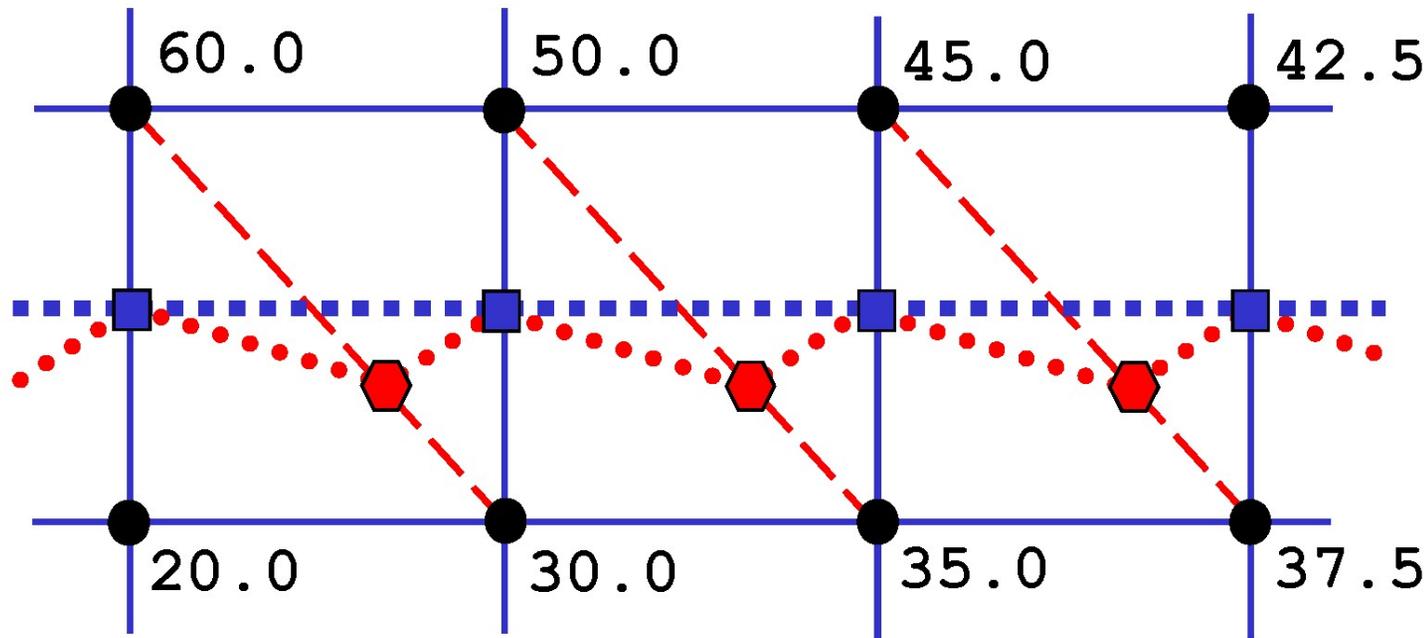   "marching squares".

Question: Why not split quadrangles into two triangles (and
   hexahedra into five or six tetrahedra) and use marching triangles
   (tetrahedra)?

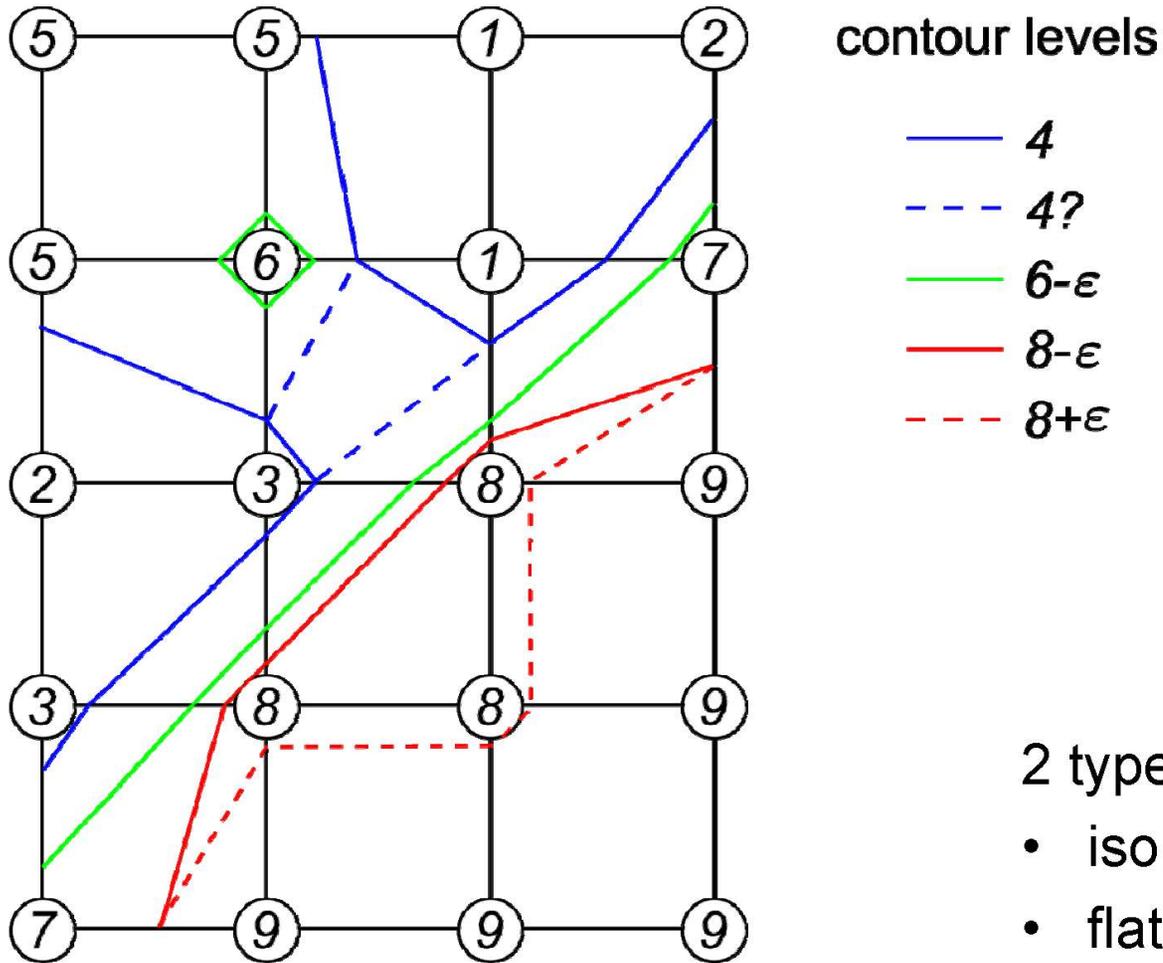Answer: This can introduce periodic artifacts!

Contours in triangle/tetrahedral cells

Illustrative example: Find contour at level *c*=40.0 !

60.0   50.0   45.0   42.5

20.0   30.0   35.0   37.5

———— original quad grid, yielding vertices ■ and contour ▪▪▪▪▪

– – – – triangulated grid,   yielding vertices ⬡ and contour ●●●●●

Example

contour levels

———	**4**
- - -	**4?**
———	**6-ε**
———	**8-ε**
- - -	**8+ε**

2 types of degeneracies:

- isolated points (*c*=6)
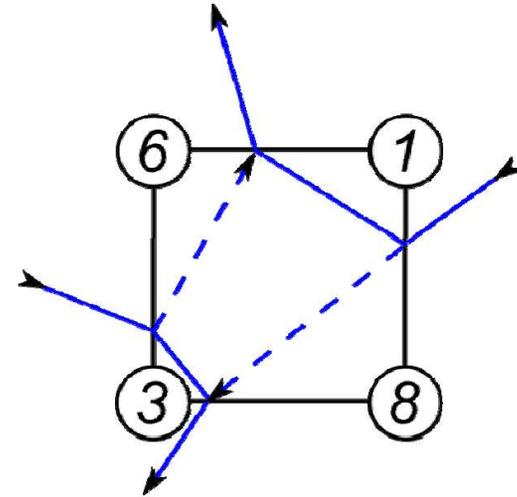- flat regions (*c*=8)

Ambiguities of contours

What is the correct contour of *c*=4?

Two possibilities, both are orientable:

- connect high values —————

- connect low values - - - - -

Answer: correctness depends on interior values of $f(x)$.

But: different interpolation schemes are possible.

Better question: What is the correct contour with respect to bilinear

  interpolation?

Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers):
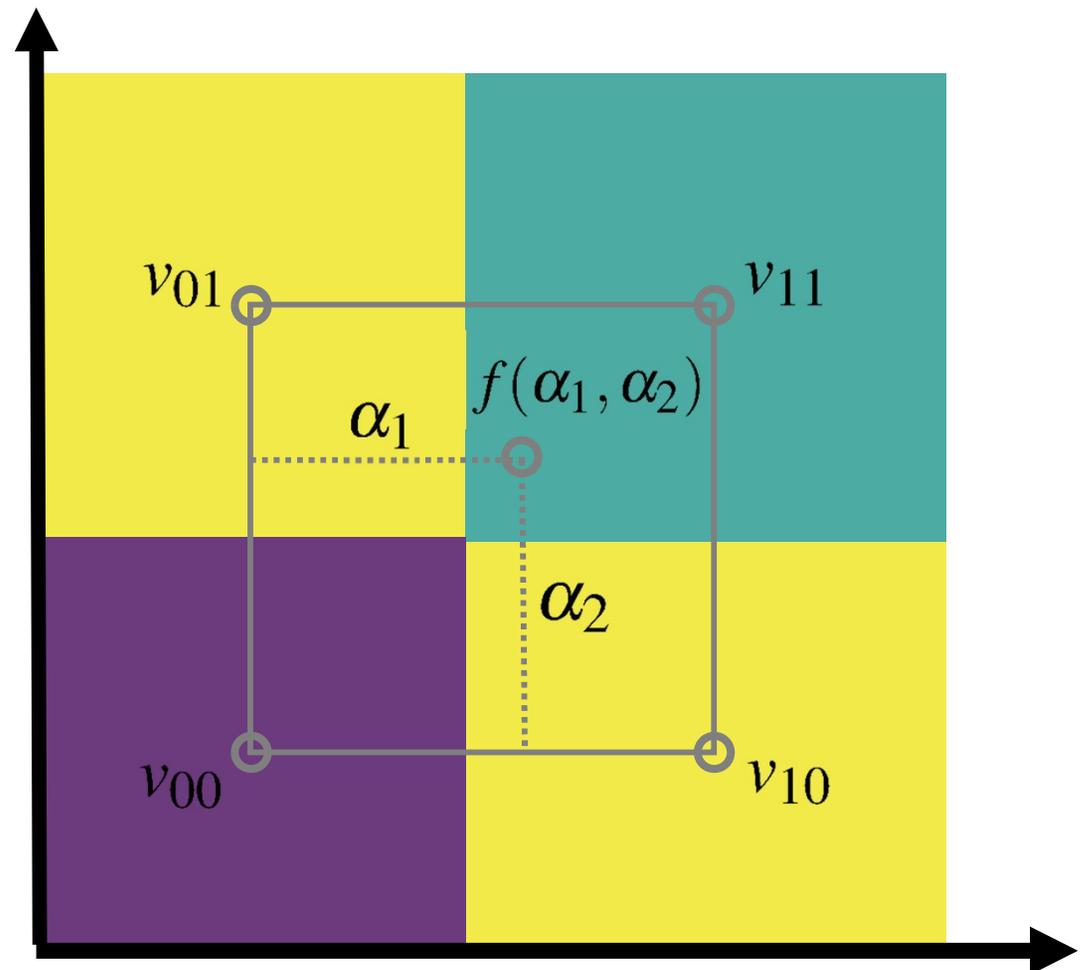
Given any (fractional) position

$$\alpha_1 := x_1 - \lfloor x_1 \rfloor \quad \alpha_1 \in [0.0, 1.0)$$
$$\alpha_2 := x_2 - \lfloor x_2 \rfloor \quad \alpha_2 \in [0.0, 1.0)$$

and 2x2 sample values

$$\begin{bmatrix} v_{01} & v_{11} \\ v_{00} & v_{10} \end{bmatrix}$$

Compute: $f(\alpha_1, \alpha_2)$

Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers):
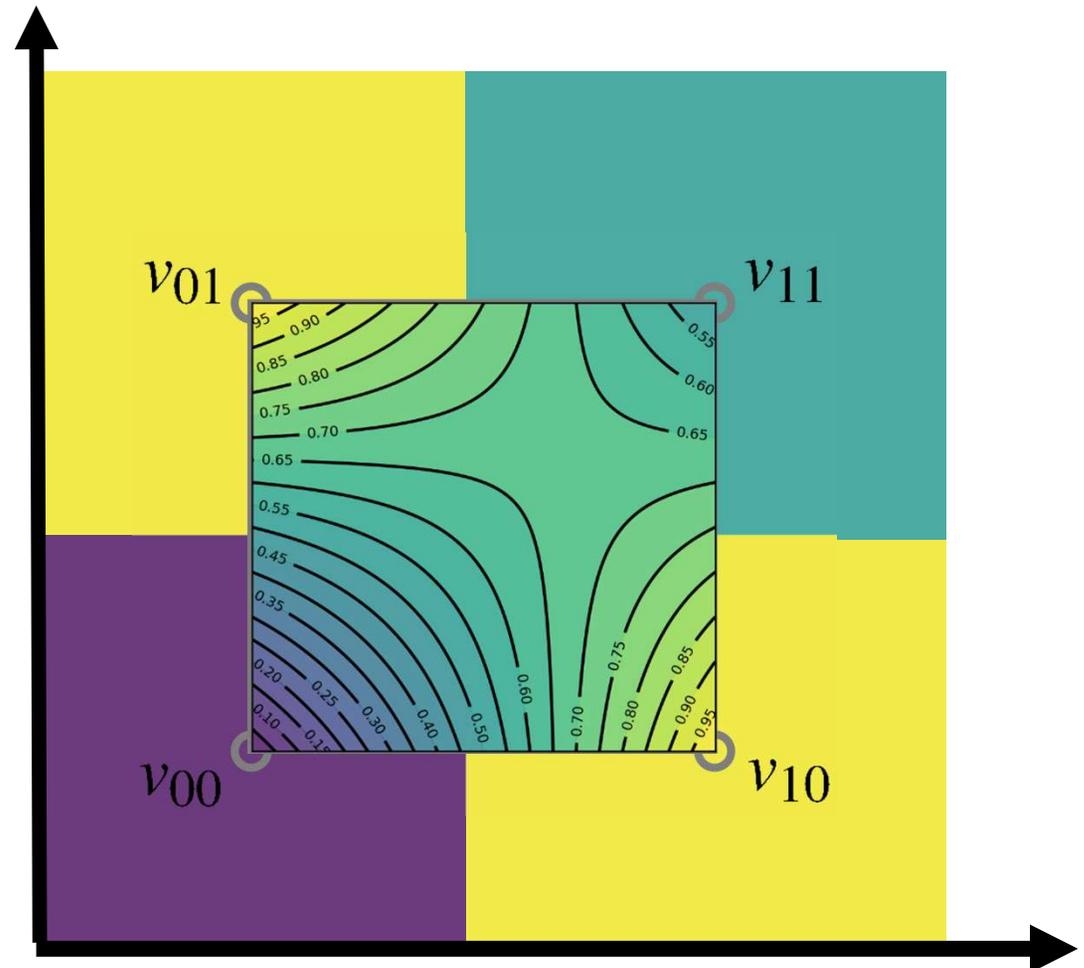
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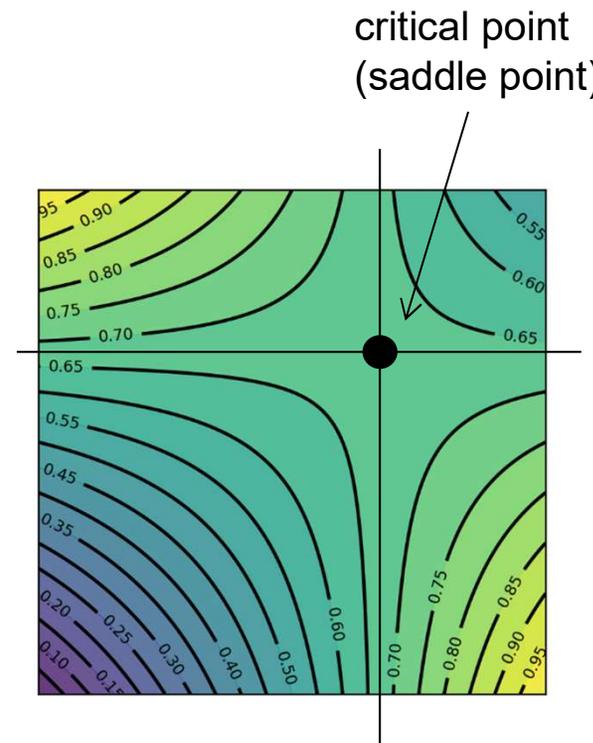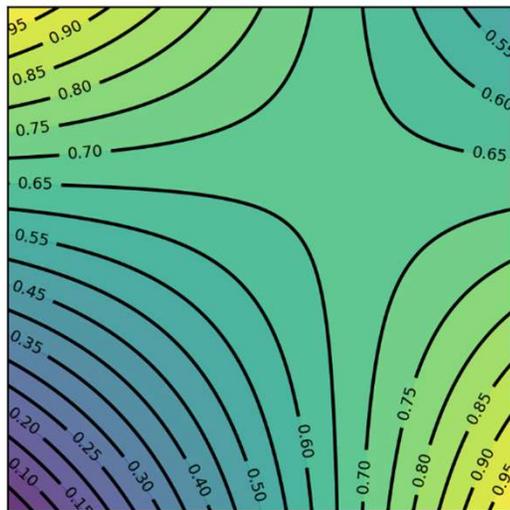
Compute: $f(\alpha_1, \alpha_2)$

Bi-Linear Interpolation: Critical Points

Critical points are where the gradient vanishes (i.e., is the zero vector)

critical point
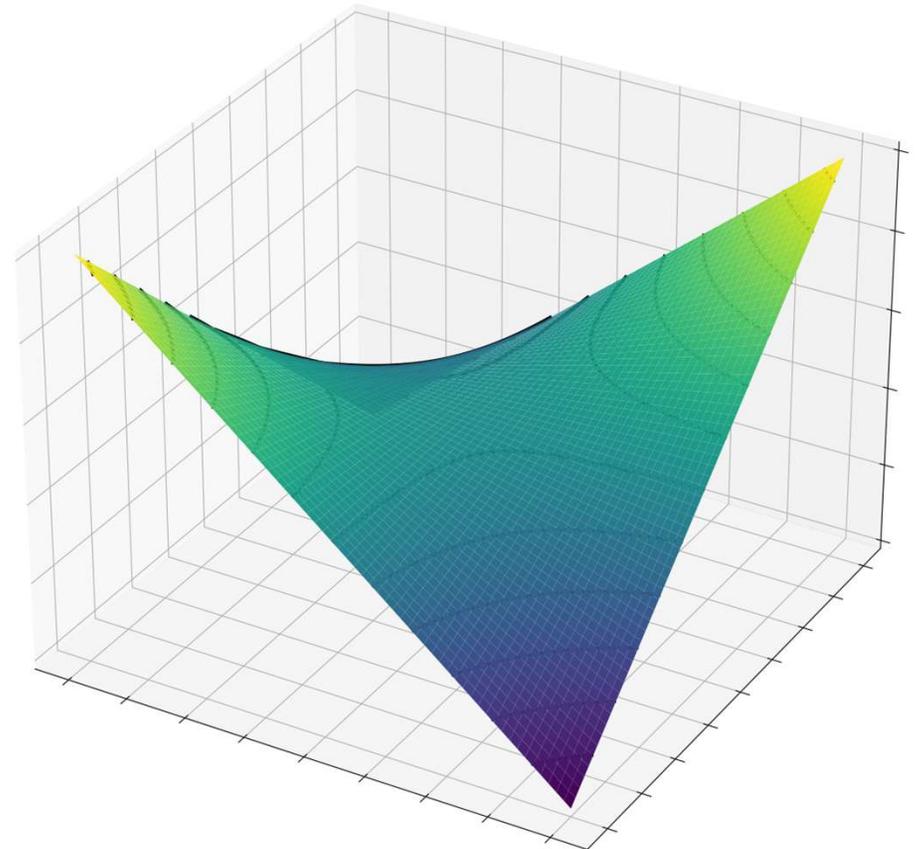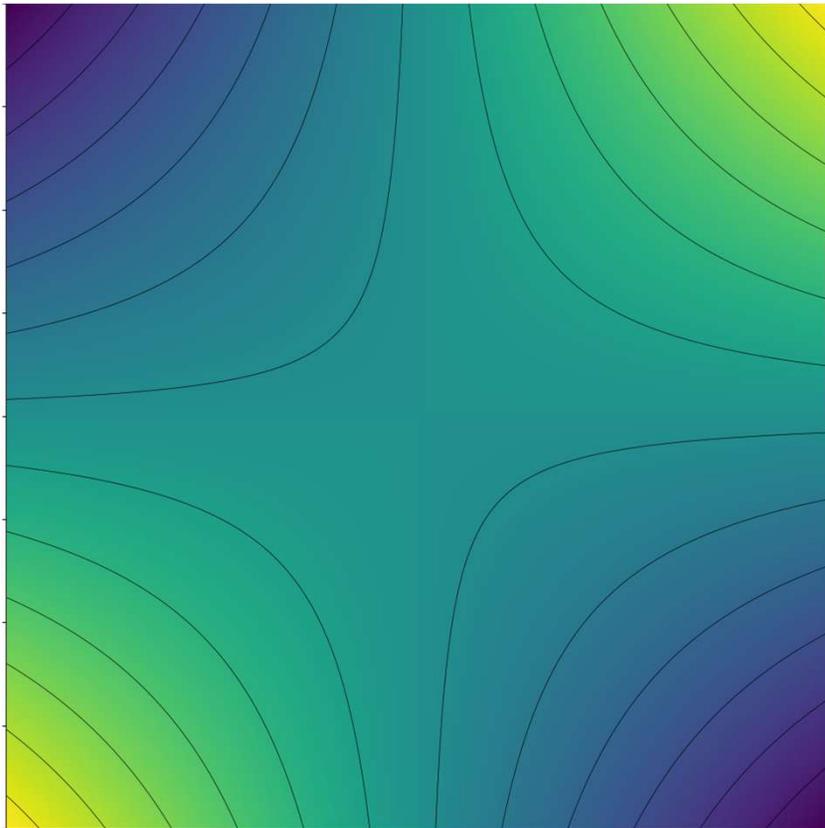(saddle point)

here, the critical
value is 2/3=0.666…

"Asymptotic decider": resolve ambiguous configurations (6 and 9) by
comparing specific iso-value with critical value (scalar value at critical point)

Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers)

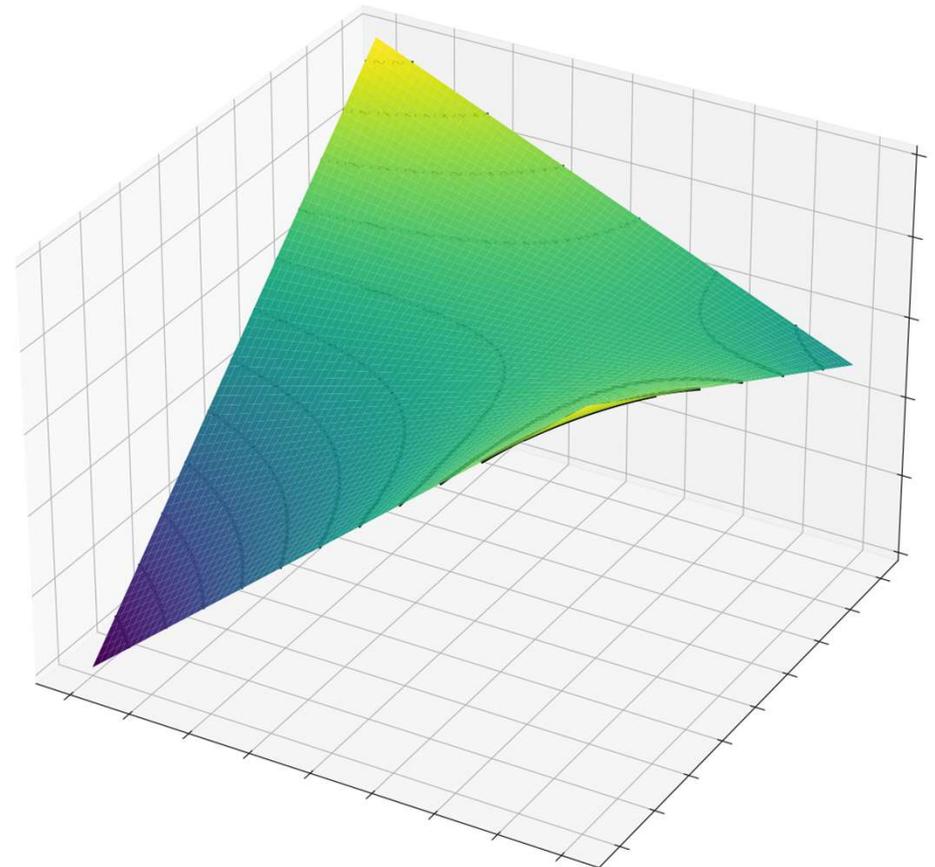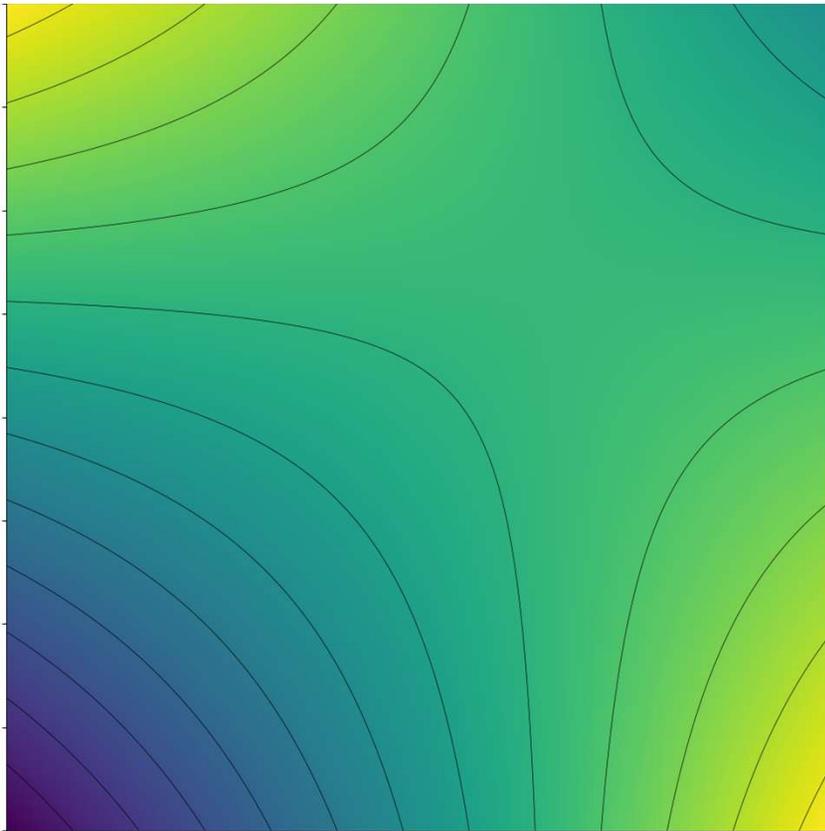Example #1: 1 at bottom-left and top-right, 0 at top-left and bottom-right

Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers)

Example #2: 1 at top-left and bottom-right, 0 at bottom-left, 0.5 at top-right

Bi-Linear Interpolation

Interpolate function at (fractional) position (α_1, α_2) :

$$f(\alpha_1, \alpha_2) = \begin{bmatrix} \alpha_2 & (1 - \alpha_2) \end{bmatrix} \begin{bmatrix} v_{01} & v_{11} \\ v_{00} & v_{10} \end{bmatrix} \begin{bmatrix} (1 - \alpha_1) \\ \alpha_1 \end{bmatrix}$$

$$= (1 - \alpha_1)(1 - \alpha_2)v_{00} + \alpha_1(1 - \alpha_2)v_{10} + (1 - \alpha_1)\alpha_2 v_{01} + \alpha_1 \alpha_2 v_{11}$$

$$= v_{00} + \alpha_1(v_{10} - v_{00}) + \alpha_2(v_{01} - v_{00}) + \alpha_1 \alpha_2(v_{00} + v_{11} - v_{10} - v_{01})$$
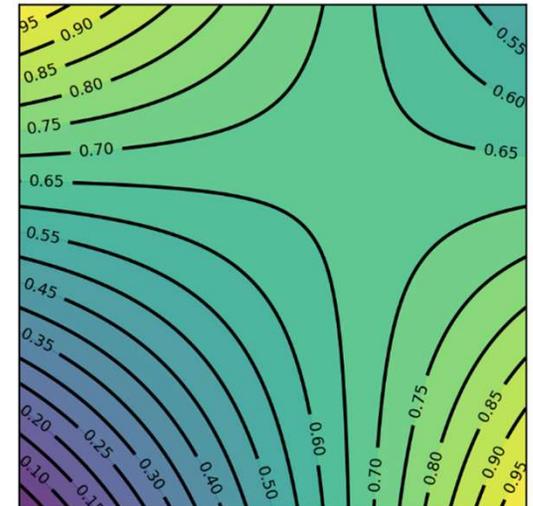
Find one specific iso-contour (can of course do this for any/all isovalues):

$$f(\alpha_1, \alpha_2) = c$$

Find all (α_1, α_2) where:

$$v_{00} + \alpha_1 (v_{10} - v_{00}) + \alpha_2 (v_{01} - v_{00}) + \alpha_1 \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01}) = c$$

Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

$$\frac{\partial f(\alpha_1, \alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1, \alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1, \alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1, \alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

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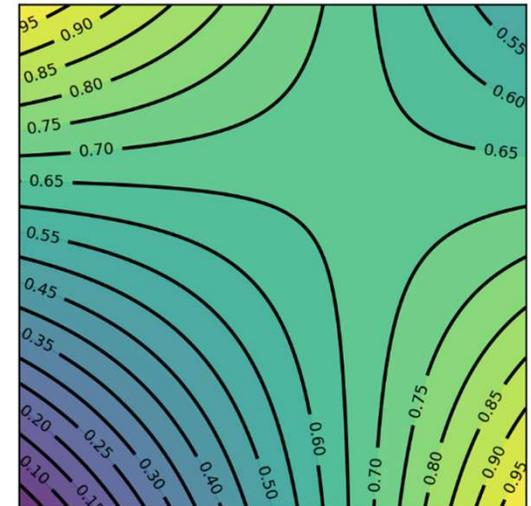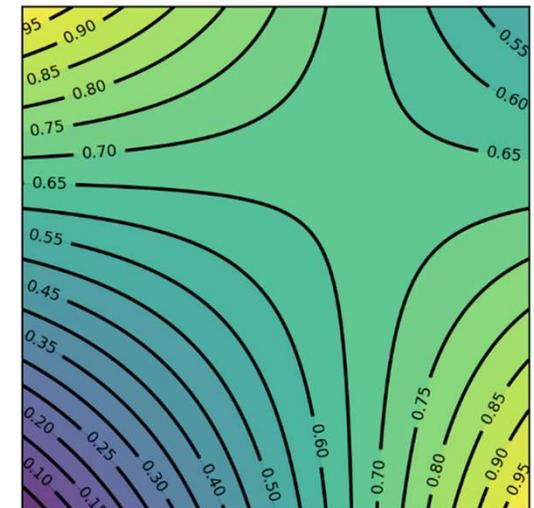Compute gradient (critical points are where gradient is zero vector):

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Where are lines of constant value / critical points?

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$$\frac{\partial f(\alpha_1, \alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

if denominator is zero, bi-linear interpolation has degenerated
to linear interpolation (or const)! (also means: no isolated critical points!)

Bi-Linear Interpolation: Critical Points

Compute gradient

Note that isolines are
    farther apart where
    gradient is smaller

Note the horizontal and
    vertical lines where
    gradient becomes
    vertical/horizontal

Note the critical point

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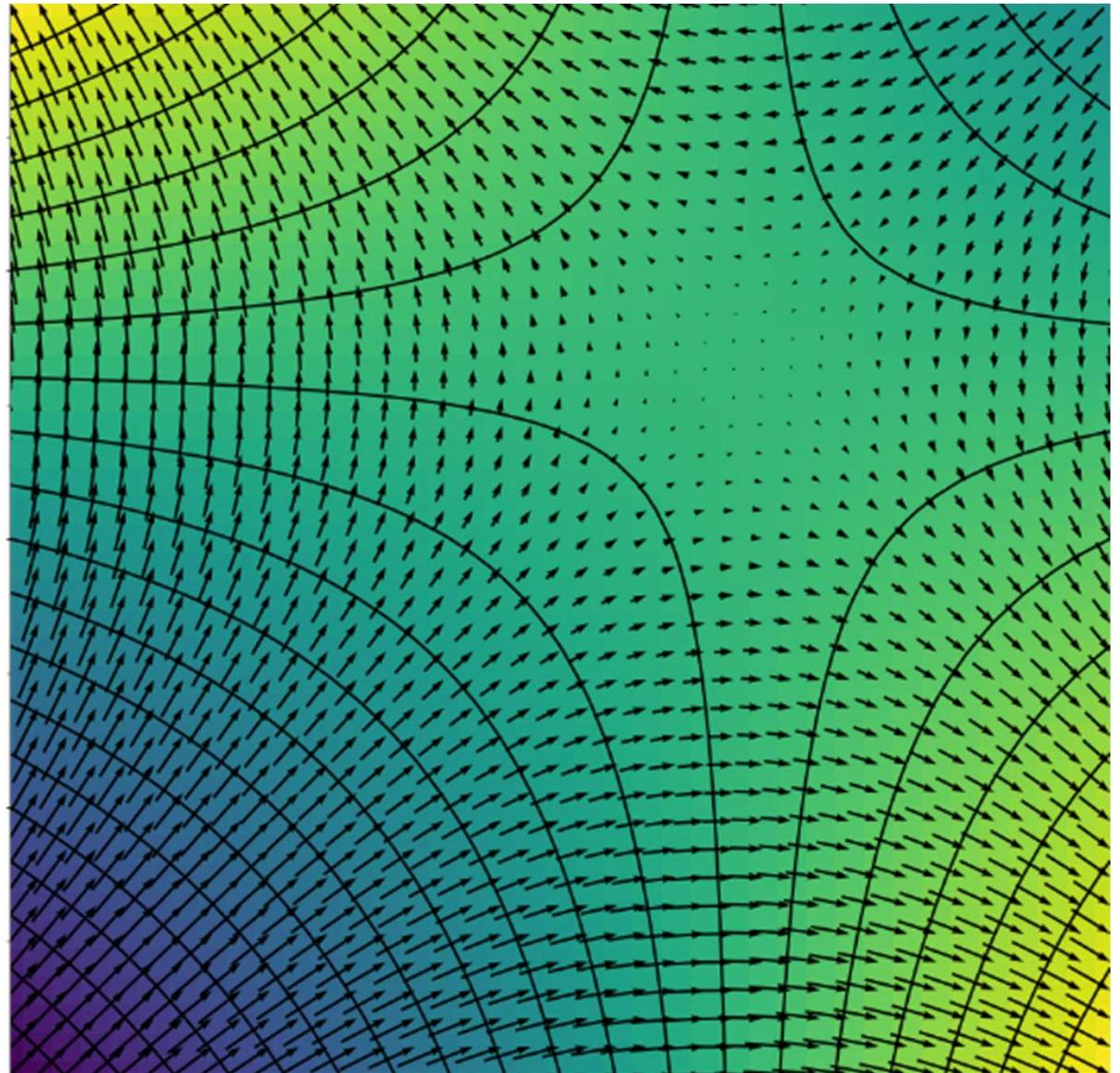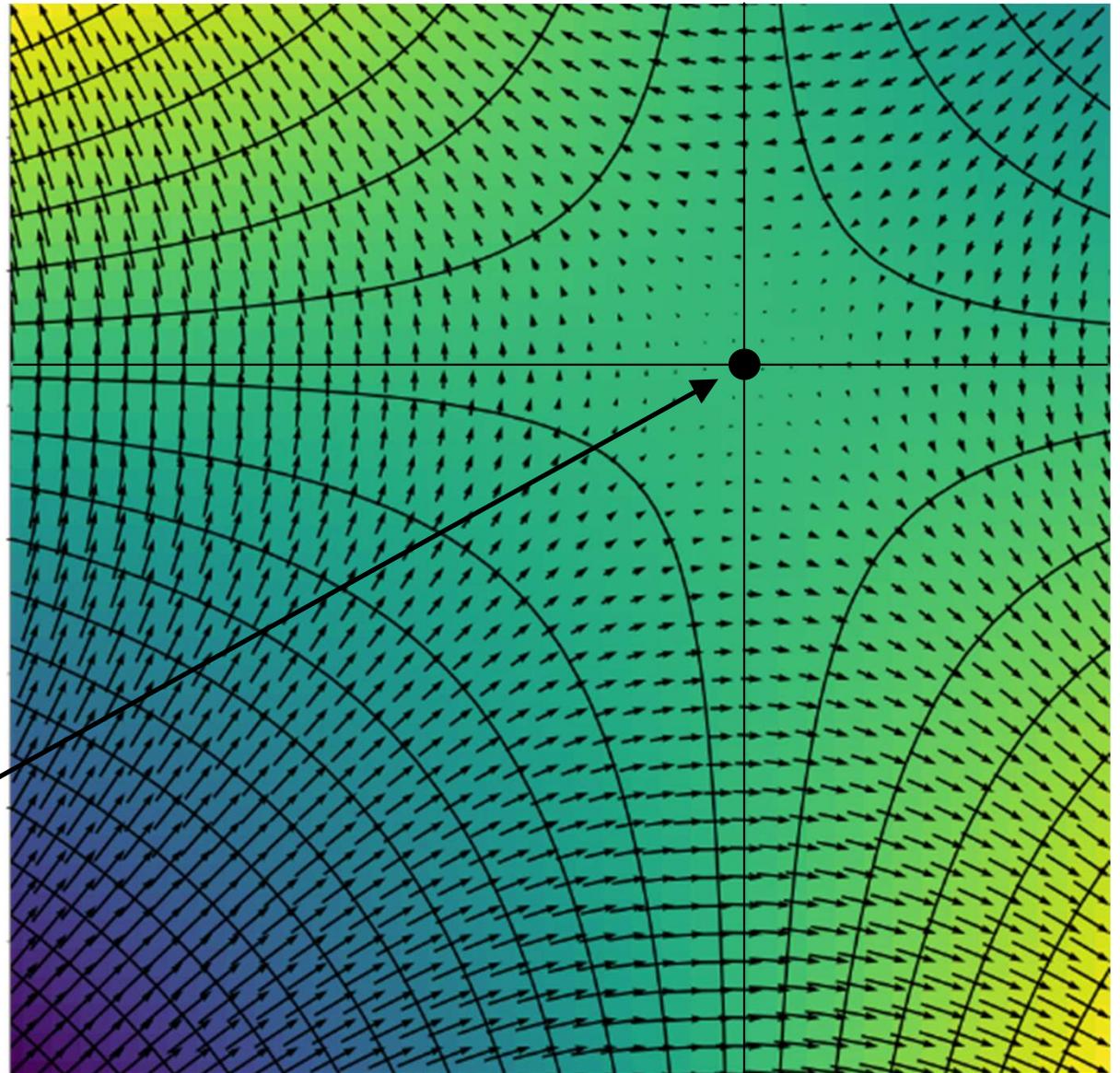# Bi-Linear Interpolation: Critical Points

Compute gradient

Note that isolines are
  farther apart where
  gradient is smaller

Note the horizontal and
  vertical lines where
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  vertical/horizontal

Note the critical point

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Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial \alpha_1^2} & \dfrac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \dfrac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \dfrac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

$$a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(here also: principal curvature magnitudes and directions
of this function's graph == surface embedded in 3D)

Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)
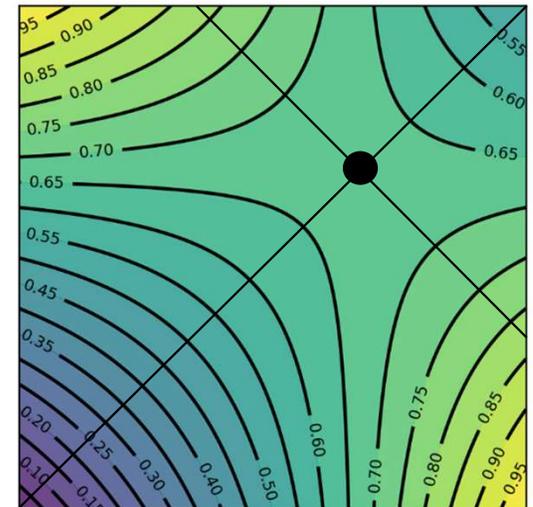
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Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)
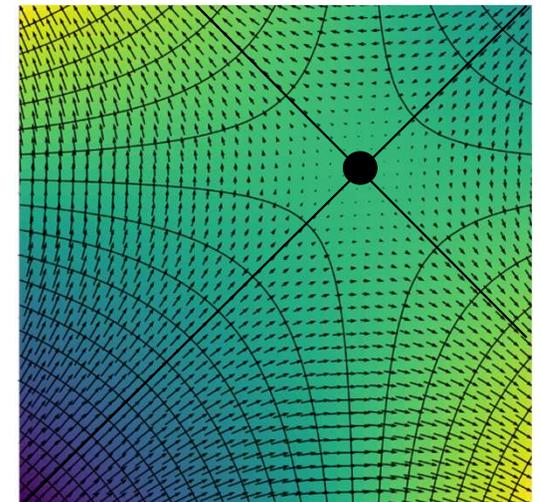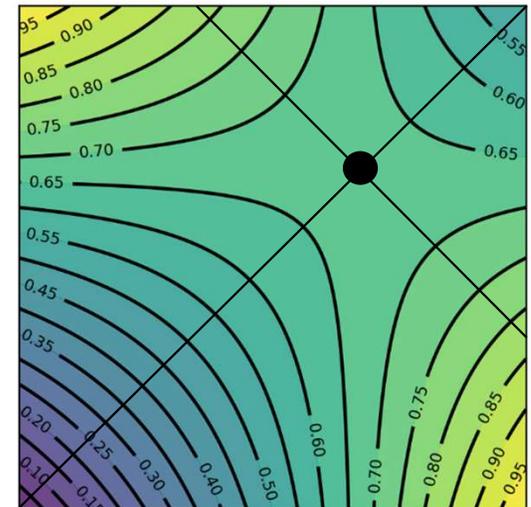
$$
\begin{bmatrix}
\dfrac{\partial^2 f}{\partial \alpha_1^2} & \dfrac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\
\dfrac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \dfrac{\partial^2 f}{\partial \alpha_2^2}
\end{bmatrix}
=
\begin{bmatrix}
0 & a \\
a & 0
\end{bmatrix}
\qquad
a = v_{00} + v_{11} - v_{10} - v_{01}
$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$
\lambda_1 = -a \text{ and } \lambda_2 = a
$$

$$
v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

degenerate means determinant = 0 (at least one eigenvalue = 0);
bi-linear is simple: a = 0 means degenerated to
linear anyway: no critical point at all! (except constant function)
(but with more than one cell: can have max or min at vertices)

When can I write an implicit function in $\mathbb{R}^{n+m}$ such that it is the graph of a function $f \colon \mathbb{R}^n \to \mathbb{R}^m$ *at least locally?*

That is: is this implicitly described function an *n*-manifold embedded in $\mathbb{R}^{n+m}$? (with local coordinates in $\mathbb{R}^n$)

$$G(f) := \{(x, f(x)) \,|\, x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^m \simeq \mathbb{R}^{n+m}$$

Theorem: if *m* x *m* Jacobian matrix is invertible

(easier for scalar field: check if gradient of f is non-zero)

See `https://en.wikipedia.org/wiki/Implicit_function_theorem`

General result: *constant rank theorem*

Thank you.

Thanks for material

- Helwig Hauser

- Eduard Gröller

- Daniel Weiskopf

- Torsten Möller

- Ronny Peikert

- Philipp Muigg

- Christof Rezk-Salama