King Abdullah University of Science and Technology

KAUST

CS 247 – Scientific Visualization
Lecture 25: Vector / Flow Visualization, Pt. 4

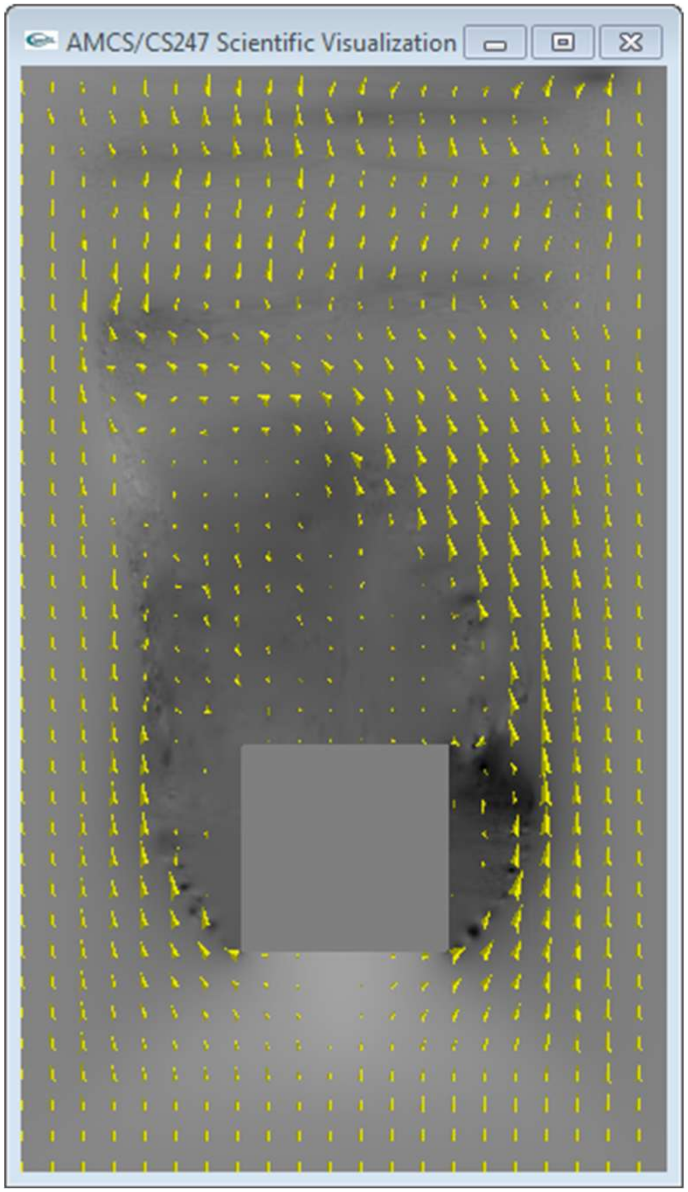
Markus Hadwiger, KAUST

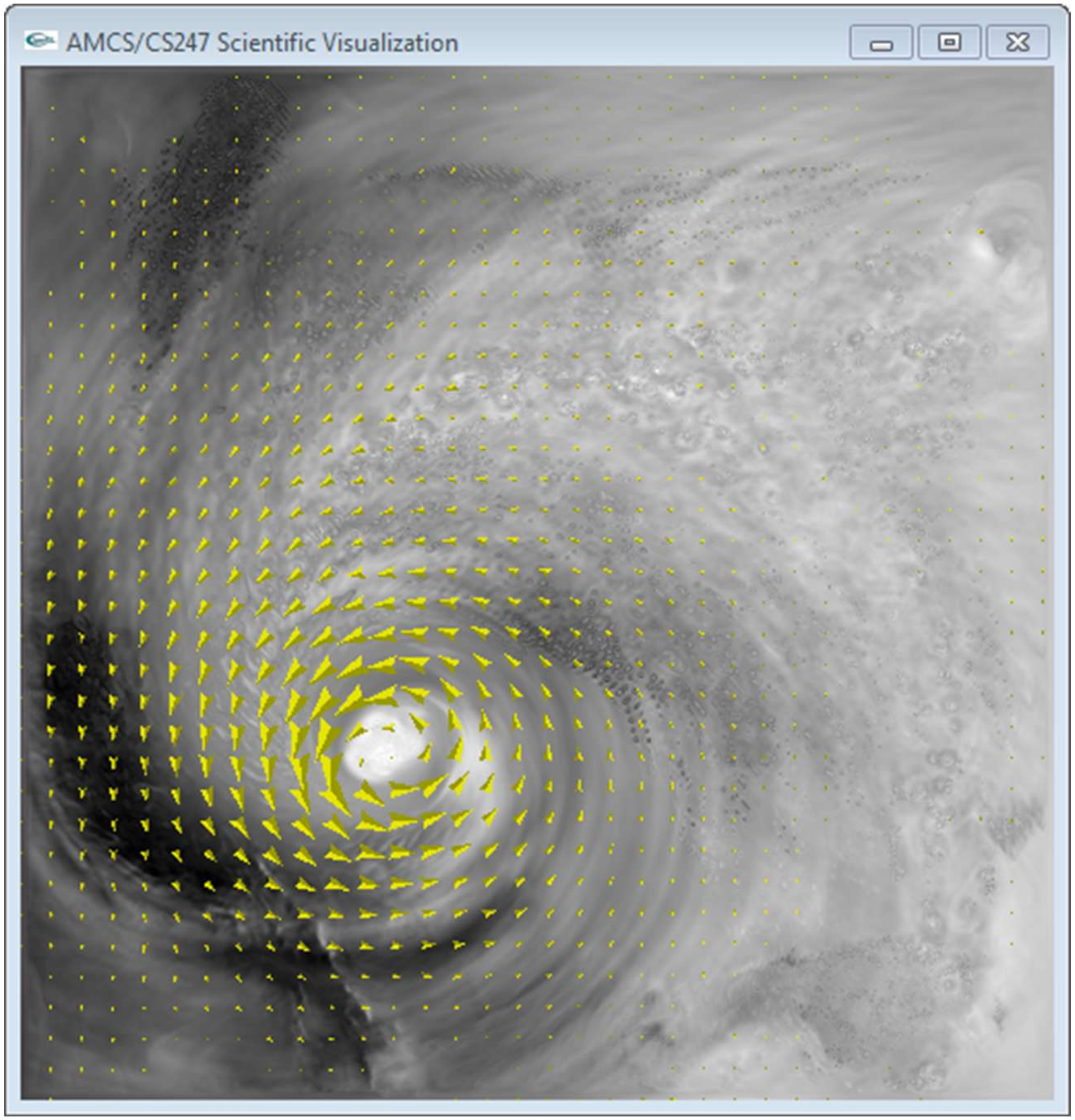
Reading Assignment #13 (until May 4)

Read (required):

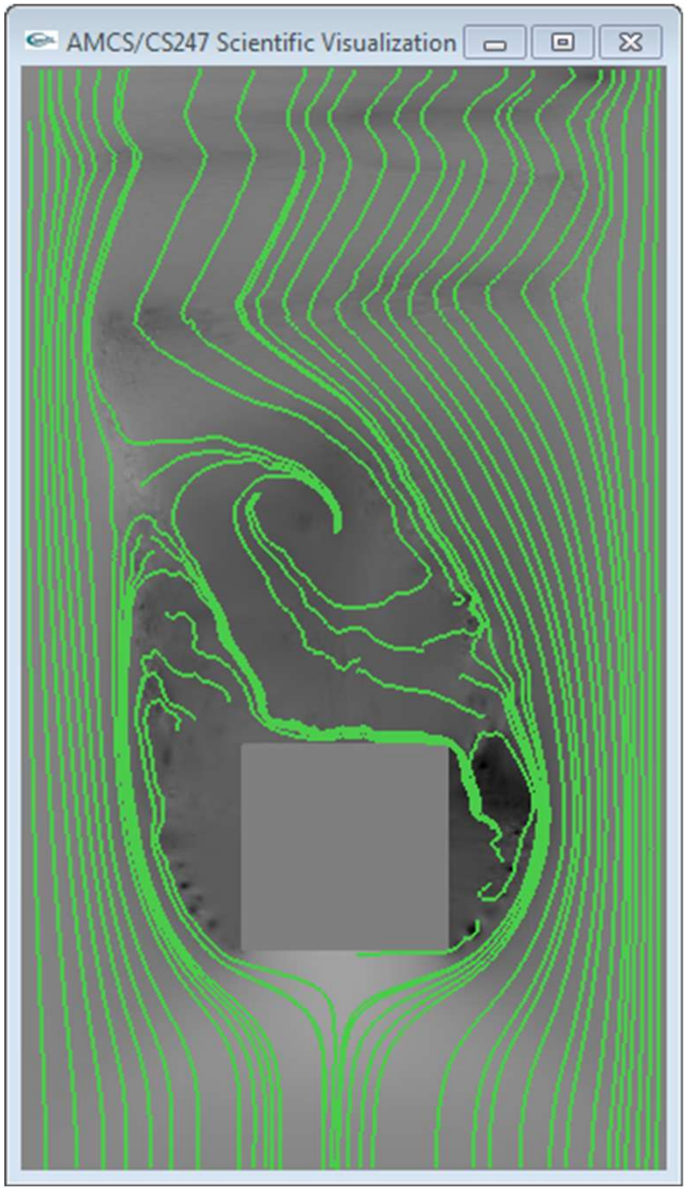
- Data Visualization book
  - Chapter 6.1 (Divergence and Vorticity)
  - Chapter 6.6 (Texture-Based Vector Visualization)
- Diffeomorphisms / smooth deformations
  `https://en.wikipedia.org/wiki/Diffeomorphism`
- Learn how convolution (the convolution of two functions) works:
  `https://en.wikipedia.org/wiki/Convolution`
- B. Cabral, C. Leedom:
  Imaging Vector Fields Using Line Integral Convolution, SIGGRAPH 1993
  `http://dx.doi.org/10.1145/166117.166151`

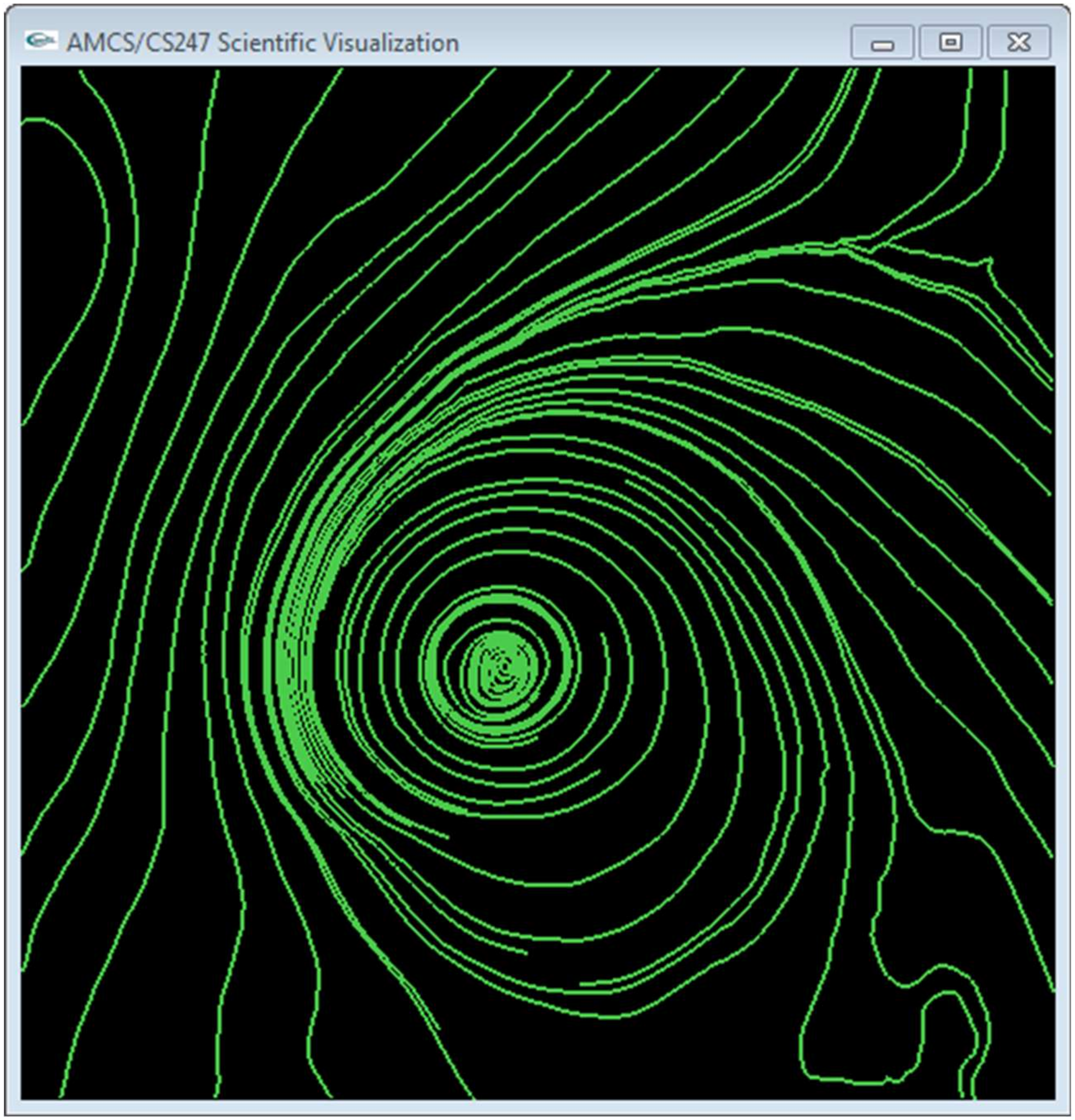
Programming Assignment #5: Flow Vis 1



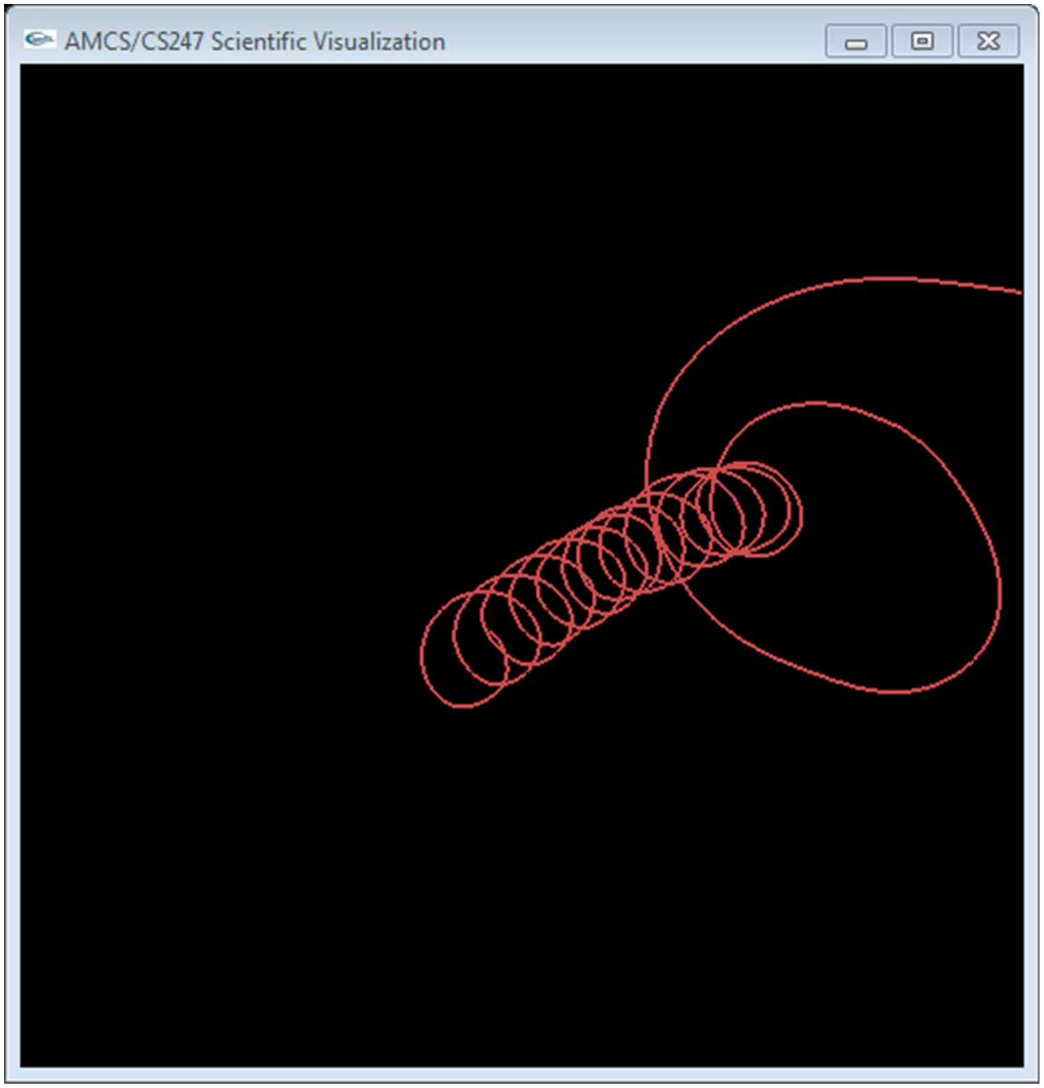


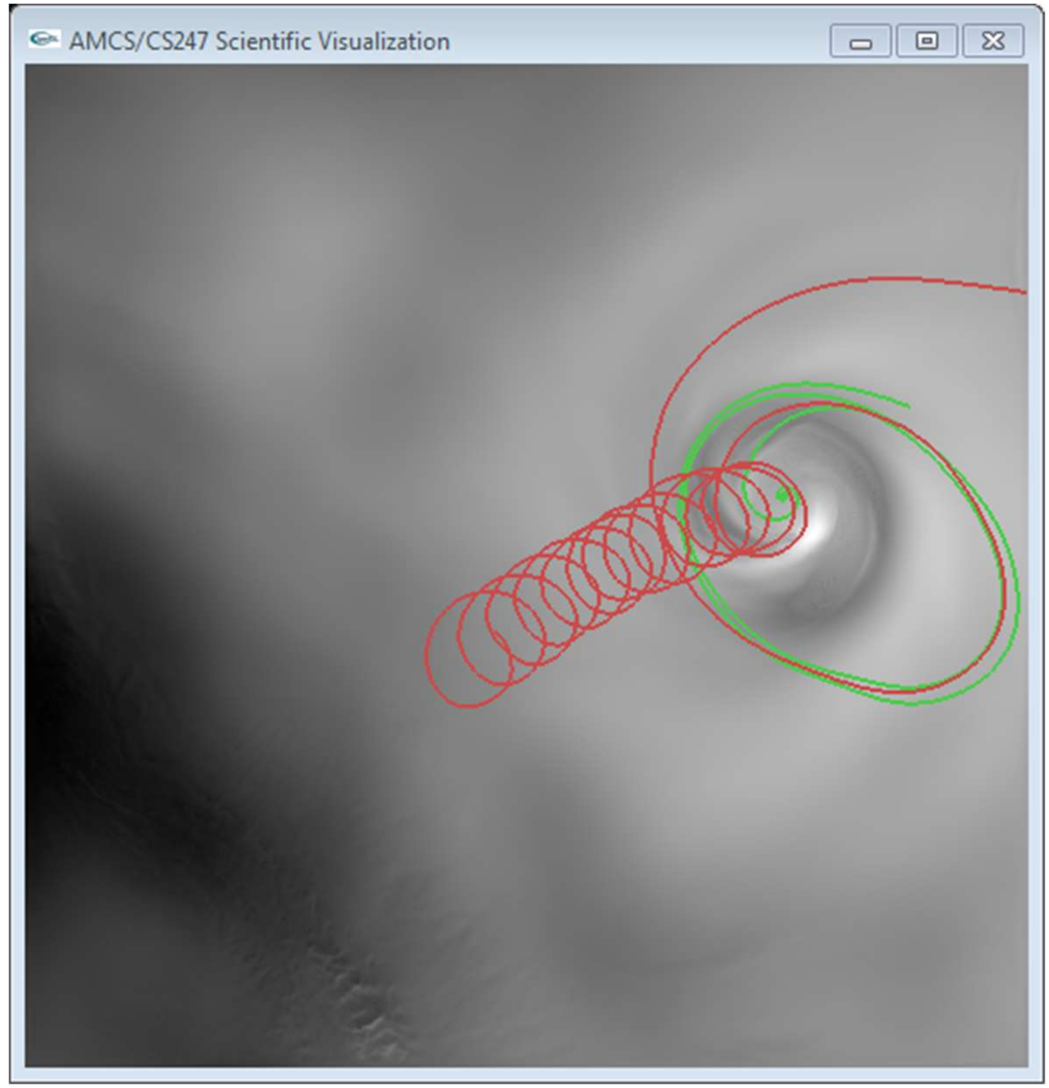
Programming Assignment #5: Flow Vis 1



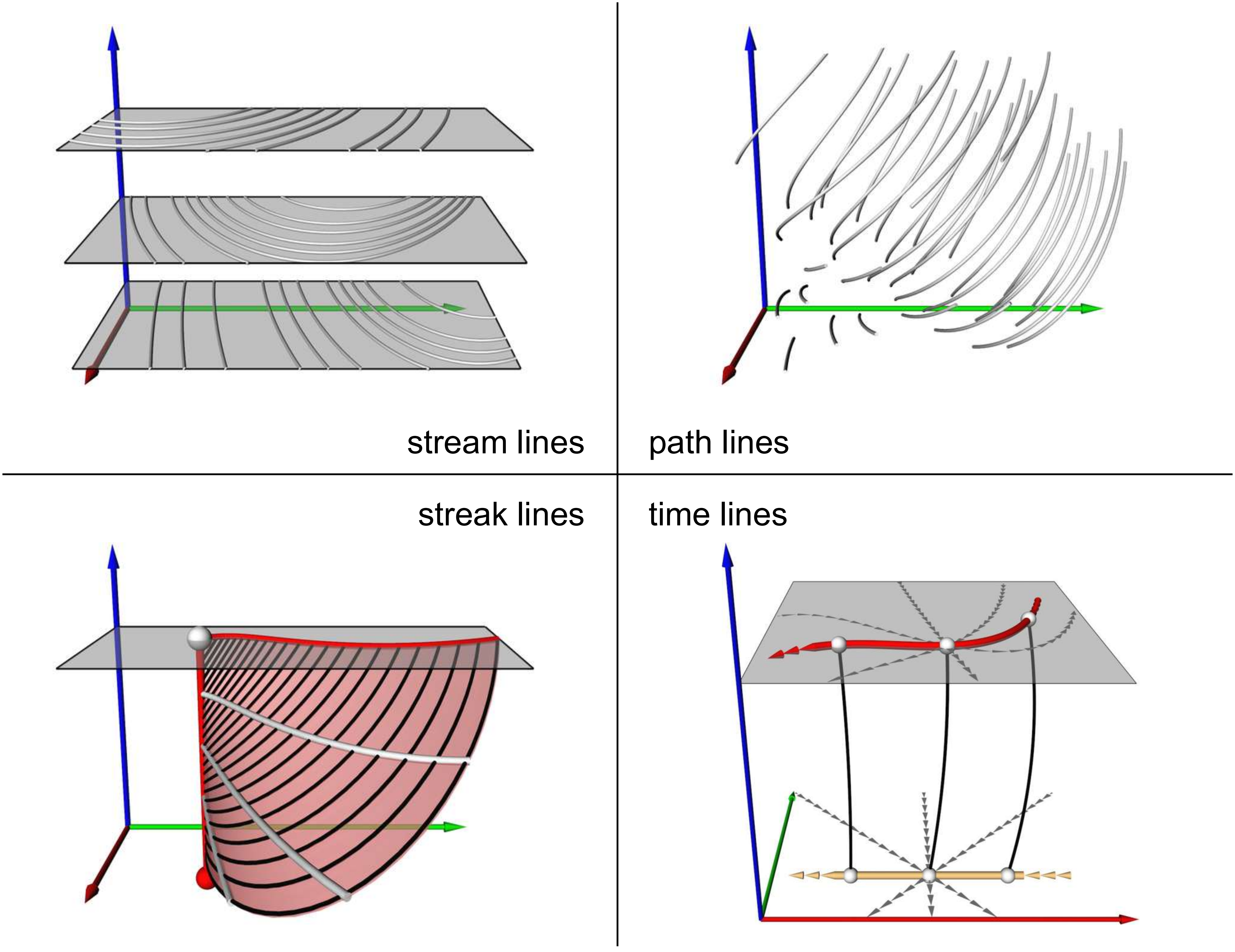


Programming Assignment #5: Flow Vis 1





Vector / Flow Visualization


stream lines
path lines
streak lines
time lines

Comparison of techniques:

(1) Pathlines:

- are physically meaningful
- allow comparison with experiment (observe marked particles)
- are well suited for dynamic visualization (of particles)

(2) Streamlines:

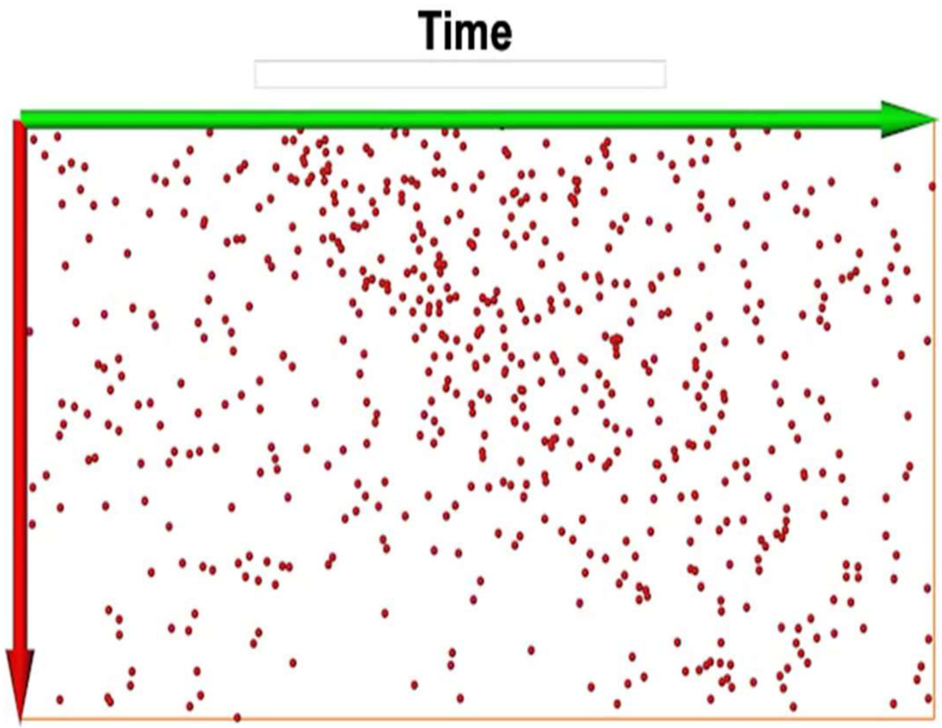
- are only geometrically, not physically meaningful
- are easiest to compute (no temporal interpolation, single IVP)
- are better suited for static visualization (prints)
- don't intersect (under reasonable assumptions)

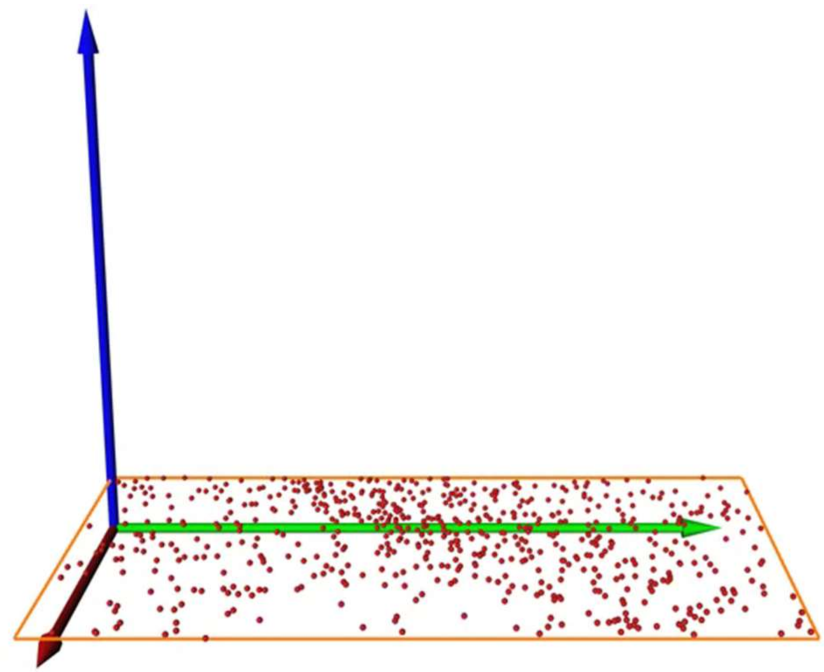
(3) Streaklines:

- are physically meaningful
- allow comparison with experiment (dye injection)
- are well suited for static and dynamic visualization
- good choice for fast moving vortices
- can be approximated by set of disconnected particles

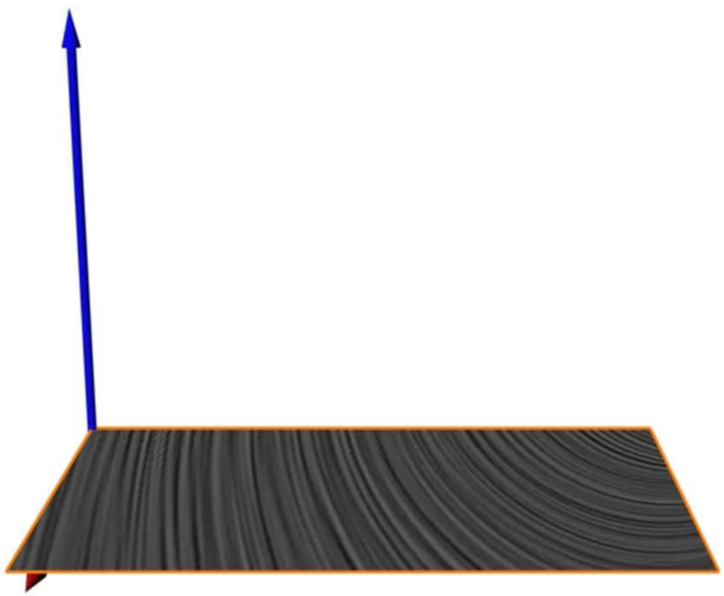
(4) Timelines:

- are physically meaningful
- are well suited for static and dynamic visualization
- can be approximated by set of disconnected particles

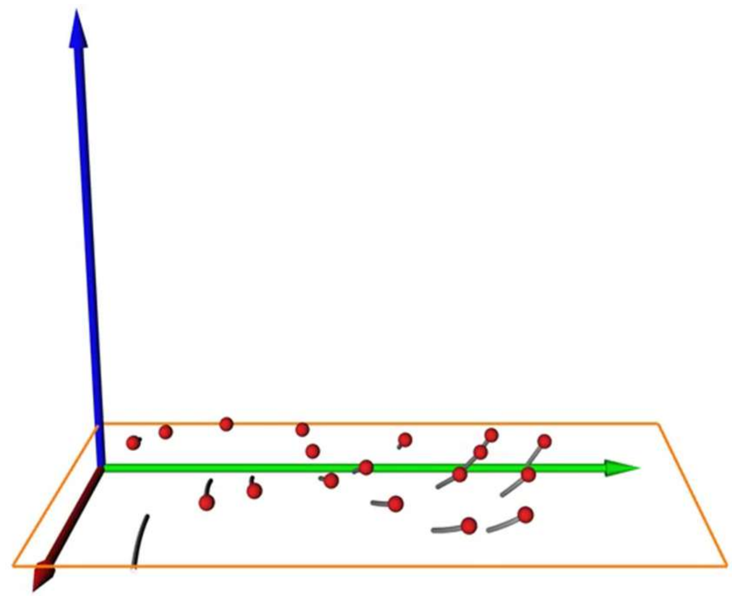




2D time-dependent vector field
particle visualization



stream lines



path lines

curve parallel to the vector field in each point for a **fixed time**

describes motion of a massless particle in an **steady** flow field

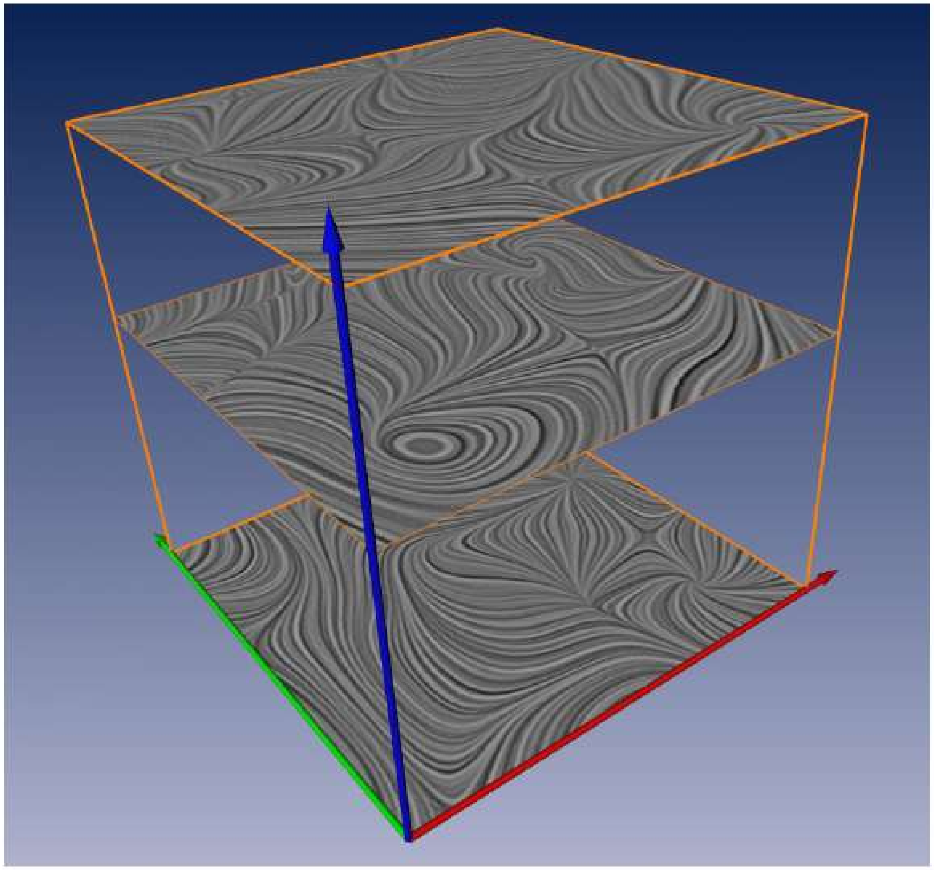
curve parallel to the vector field in each point **over time**

describes motion of a massless particle in an **unsteady** flow field

Streamlines Over Time

Defined only for steady flow or for a fixed time step (of unsteady flow)

Different tangent curves in every time step for time-dependent vector fields (unsteady flow)

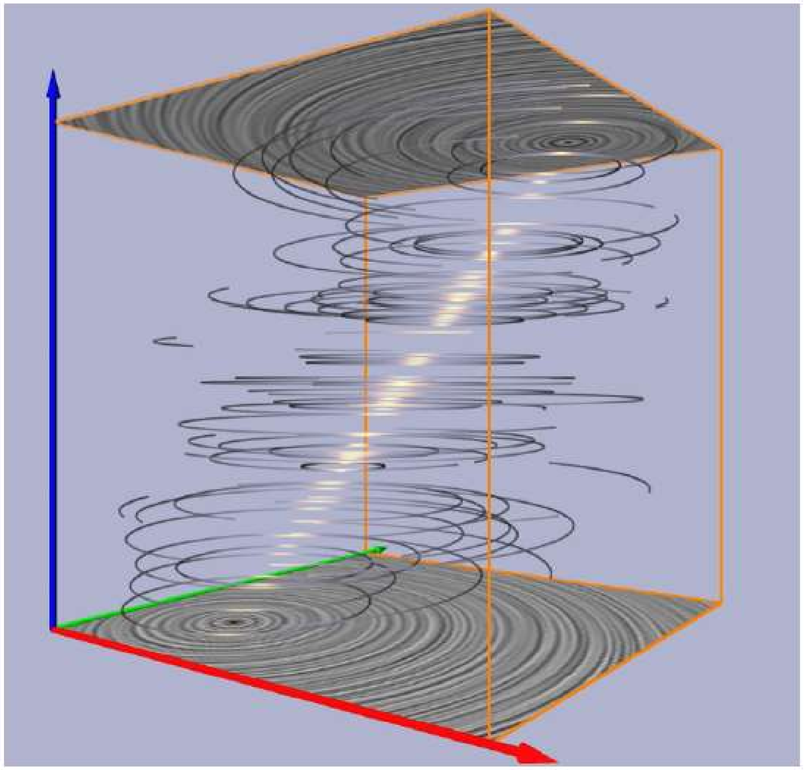


Tino Weinkauf

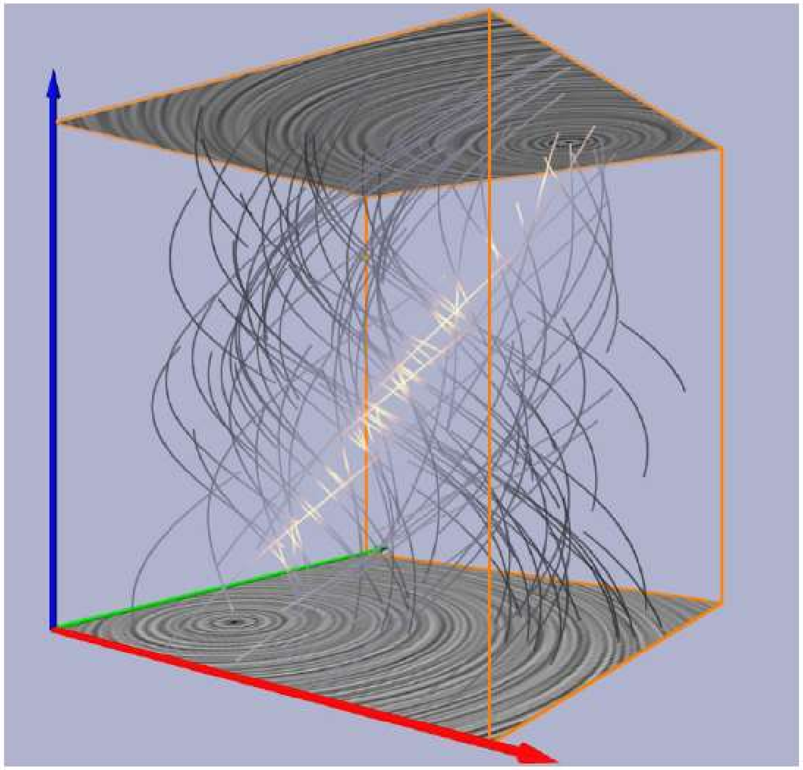
Stream Lines vs. Path Lines Viewed Over Time

Plotted with time as third dimension

- Tangent curves to a (n + 1)-dimensional vector field



Stream Lines



Path Lines

Vector fields as ODEs

For simplicity, the vector field is now interpreted as a velocity field.

Then the field $\mathbf{v}(\mathbf{x},t)$ describes the connection between location and velocity of a (massless) particle.

It can equivalently be expressed as an ordinary differential equation

$$\dot{\mathbf{x}}(t)=\mathbf{v}(\mathbf{x}(t),t)$$

This ODE, together with an initial condition

$$\mathbf{x}(t_0)=\mathbf{x}_0\ ,$$

is a so-called initial value problem (IVP).

Its solution is the integral curve (or trajectory)

$$\mathbf{x}(t)=\mathbf{x}_0+\int_{t_0}^{t}\mathbf{v}(\mathbf{x}(\tau),\tau)\,d\tau$$

The integral curve is a pathline, describing the path of a massless particle which was released at time t_0 at position x_0.

Remark: $t < t_0$ is allowed.

For static fields, the ODE is autonomous:

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t))$$

and its integral curves

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(\tau))\, d\tau$$

are called field lines, or (in the case of velocity fields) streamlines.

In static vector fields, pathlines and streamlines are identical.

In time-dependent vector fields, instantaneous streamlines can be computed from a "snapshot" at a fixed time T (which is a static vector field)

$$\mathbf{v}_T(\mathbf{x}) = \mathbf{v}(\mathbf{x}, T)$$

In practice, time-dependent fields are often given as a dataset per time step. Each dataset is then a snapshot.

Streamline integration

Outline of algorithm for numerical streamline integration (with obvious extension to pathlines):

Inputs:

- static vector field $\mathbf{v}(\mathbf{x})$
- seed points with time of release $(\mathbf{x}_0, t_0)$
- control parameters:
  - step size (temporal, spatial, or in local coordinates)
  - step count limit, time limit, etc.
  - order of integration scheme

Output:

- streamlines as "polylines", with possible attributes (interpolated field values, time, speed, arc length, etc.)

Preprocessing:

- set up search structure for point location
- for each seed point:
  - global point location: Given a point **x**,
    find the cell containing **x** and the local coordinates (ξ, η, ζ)
    or ir the grid is structured:
    find the computational space coordinates $(i+\xi,\ j+\eta,\ k+\zeta)$
  - If **x** is not found in a cell, remove seed point

Integration loop, for each seed point **x**:

- interpolate ***v*** trilinearly to local coordinates (ξ, η, ζ)
- do an integration step, producing a new point **x'**
- incremental point location: For position **x'** find cell and local coordinates (ξ', η', ζ') making use of information (coordinates, local coordinates, cell) of old point **x**

Termination criteria:

- grid boundary reached
- step count limit reached
- optional: velocity close to zero
- optional: time limit reached
- optional: arc length limit reached

Integration step: widely used integration methods:

- Euler (used only in special speed-optimized techniques, e.g. GPU-based texture advection)

$$\mathbf{x}_{new} = \mathbf{x} + \mathbf{v}(\mathbf{x}, t) \cdot \triangle t$$

- Runge-Kutta, 2nd or 4th order

Higher order than 4th?

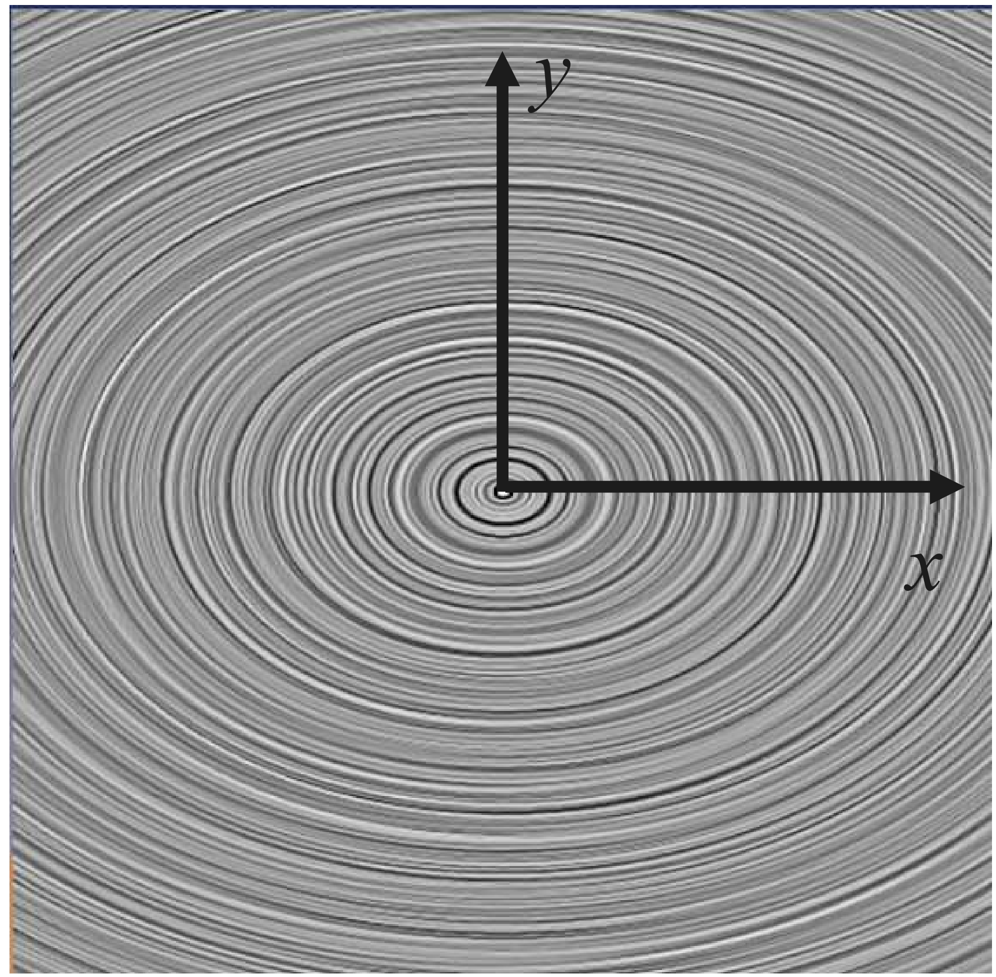
- often too slow for visualization
- study (Yeung/Pope 1987) shows that, when using standard trilinear interpolation, interpolation errors dominate integration errors.

Numerical Integration

- **Numerical integration of stream lines:**

- approximate streamline by polygon $\mathbf{x}_i$

- **Testing example:**
  - $\mathbf{v}(x,y) = (-y, x/2)$^$T$
  - exact solution: ellipses
  - starting integration from (0,-1)

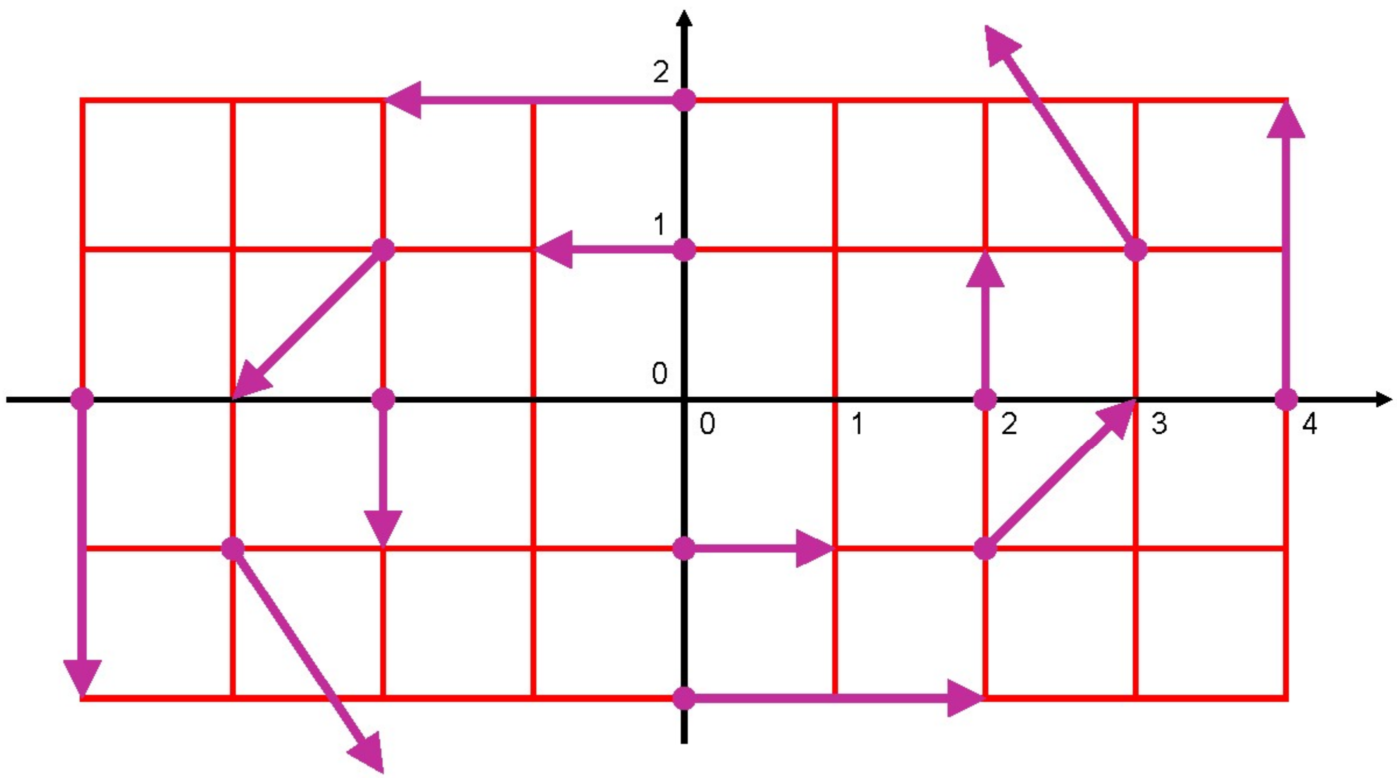


Streamlines – Practice

- Basic approach:
  - theory: $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{s}(u))\, \mathrm{d}u$
  - practice: numerical integration
  - idea:
    (very) locally, the solution is (approx.) linear
  - Euler integration:
    follow the current flow vector $\mathbf{v}(\mathbf{s}_i)$ from the current streamline point $\mathbf{s}_i$ for a very small time ($\mathrm{d}t$) and therefore distance
  - Euler integration: $\mathbf{s}_{i+1} = \mathbf{s}_i + \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i)$,
    integration of small steps ($\mathrm{d}t$ very small)

Euler Integration – Example

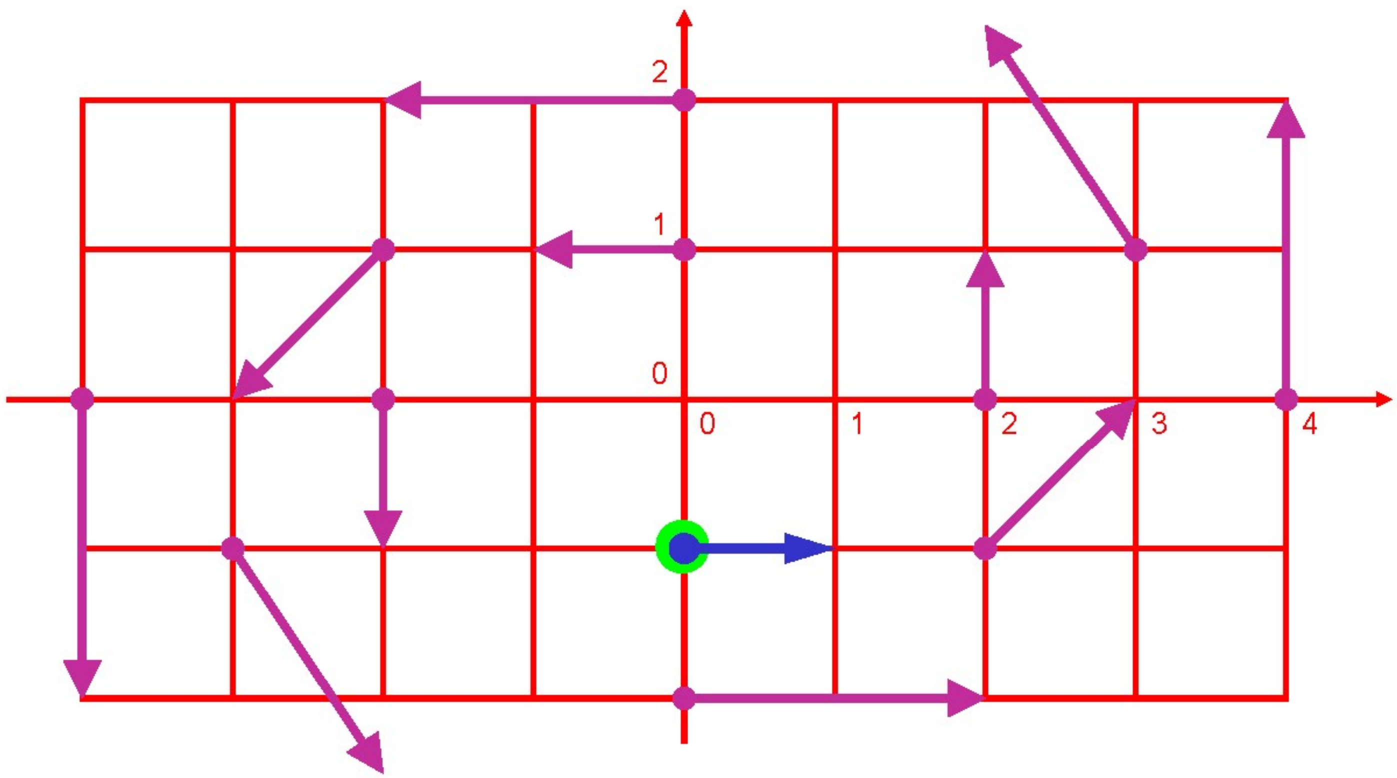
- 2D model data: $v_x = dx/dt = -y$
  $v_y = dy/dt = x/2$
- Sample arrows:



- True solution: ellipses!

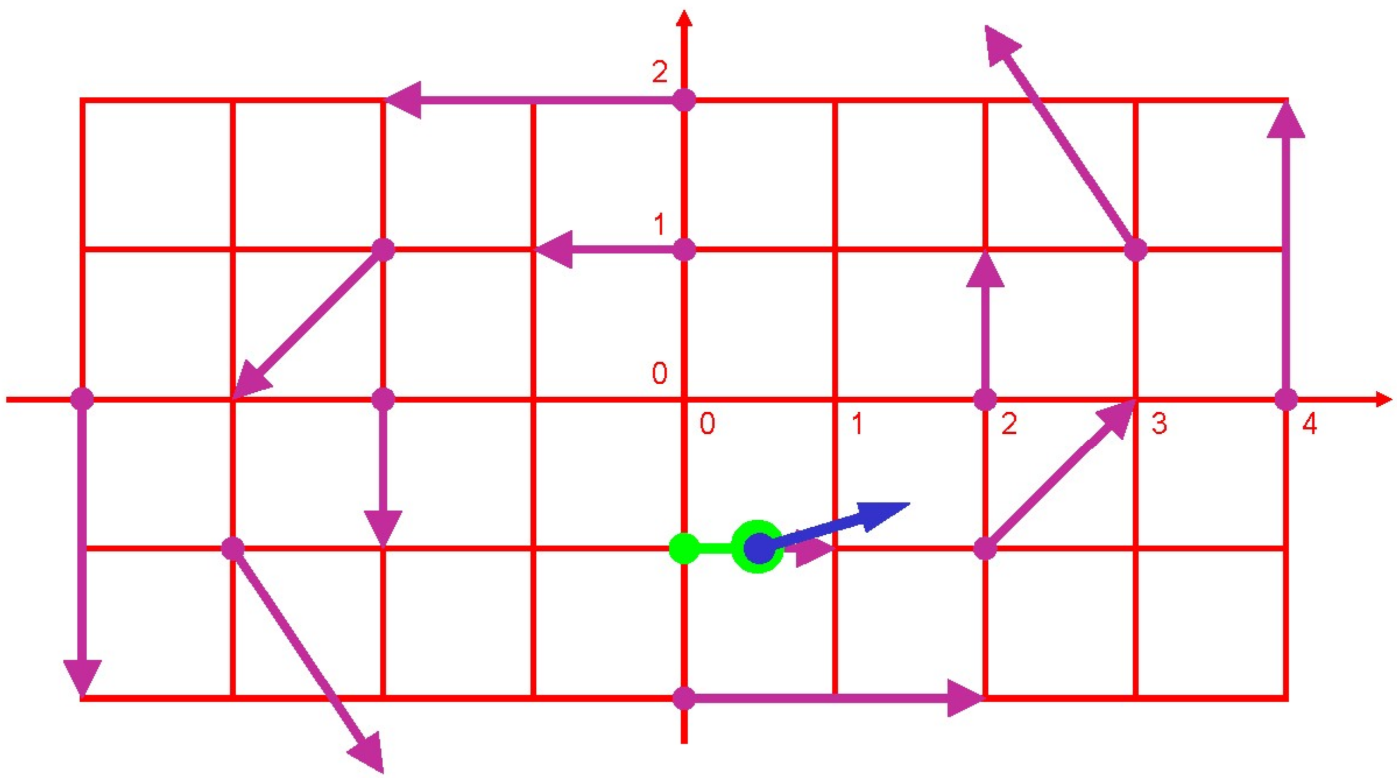
Euler Integration – Example

- Seed point $\mathbf{s}_0 = (0|-1)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_0) = (1|0)^T$;
  dt = 1/2



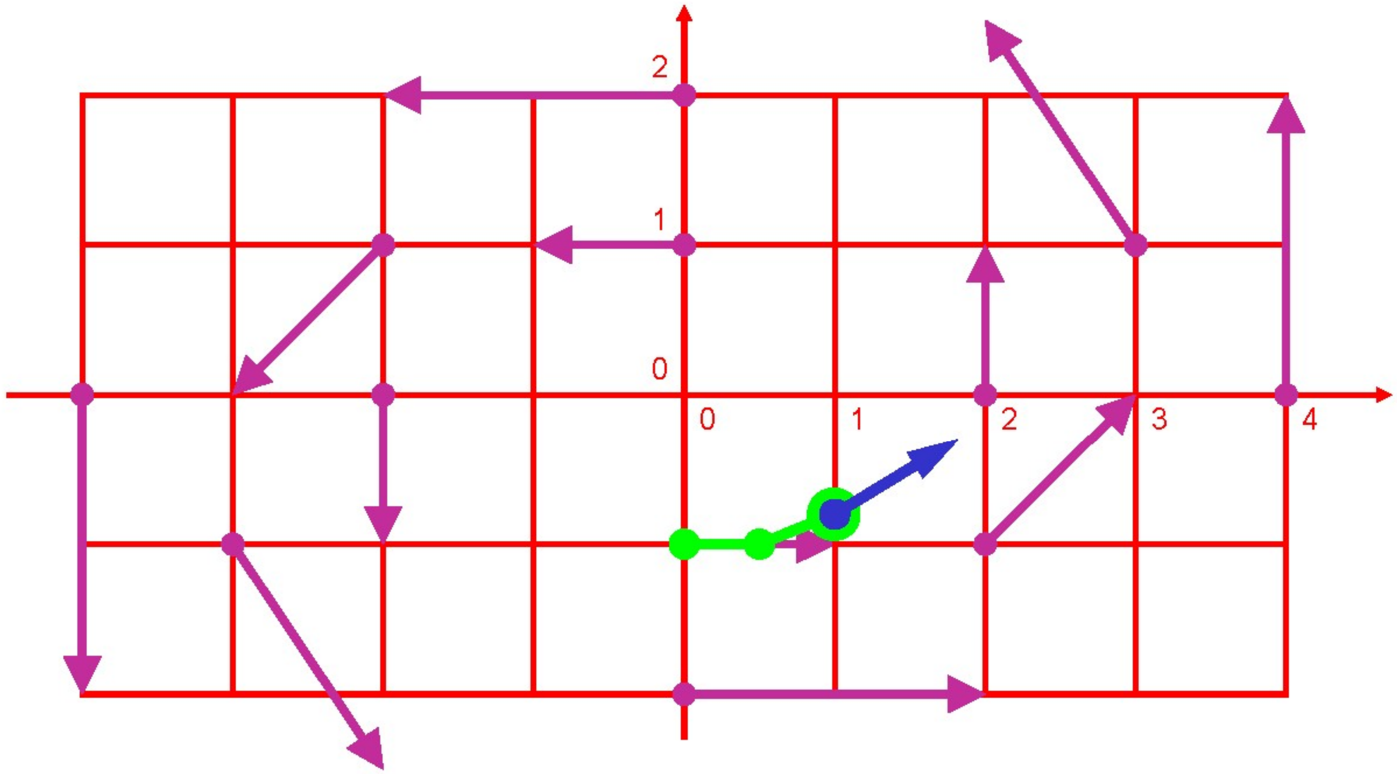
Euler Integration – Example

- New point $\mathbf{s}_1 = \mathbf{s}_0 + \mathbf{v}(\mathbf{s}_0) \cdot dt = (1/2\,|\,-1)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_1) = (1\,|\,1/4)^T$;



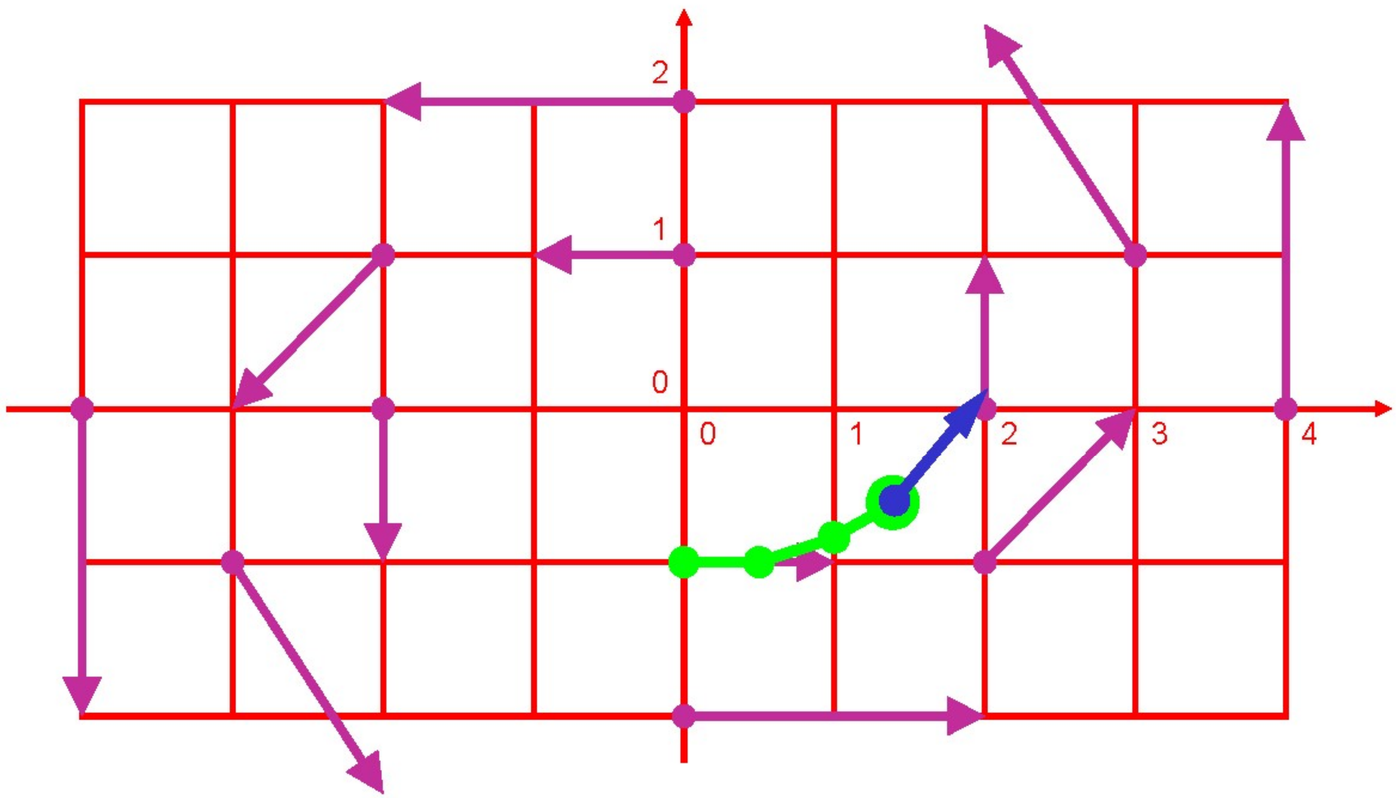
Euler Integration – Example

- New point $\mathbf{s}_2 = \mathbf{s}_1 + \mathbf{v}(\mathbf{s}_1) \cdot dt = (1\,|\,-7/8)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_2) = (7/8\,|\,1/2)^T$;



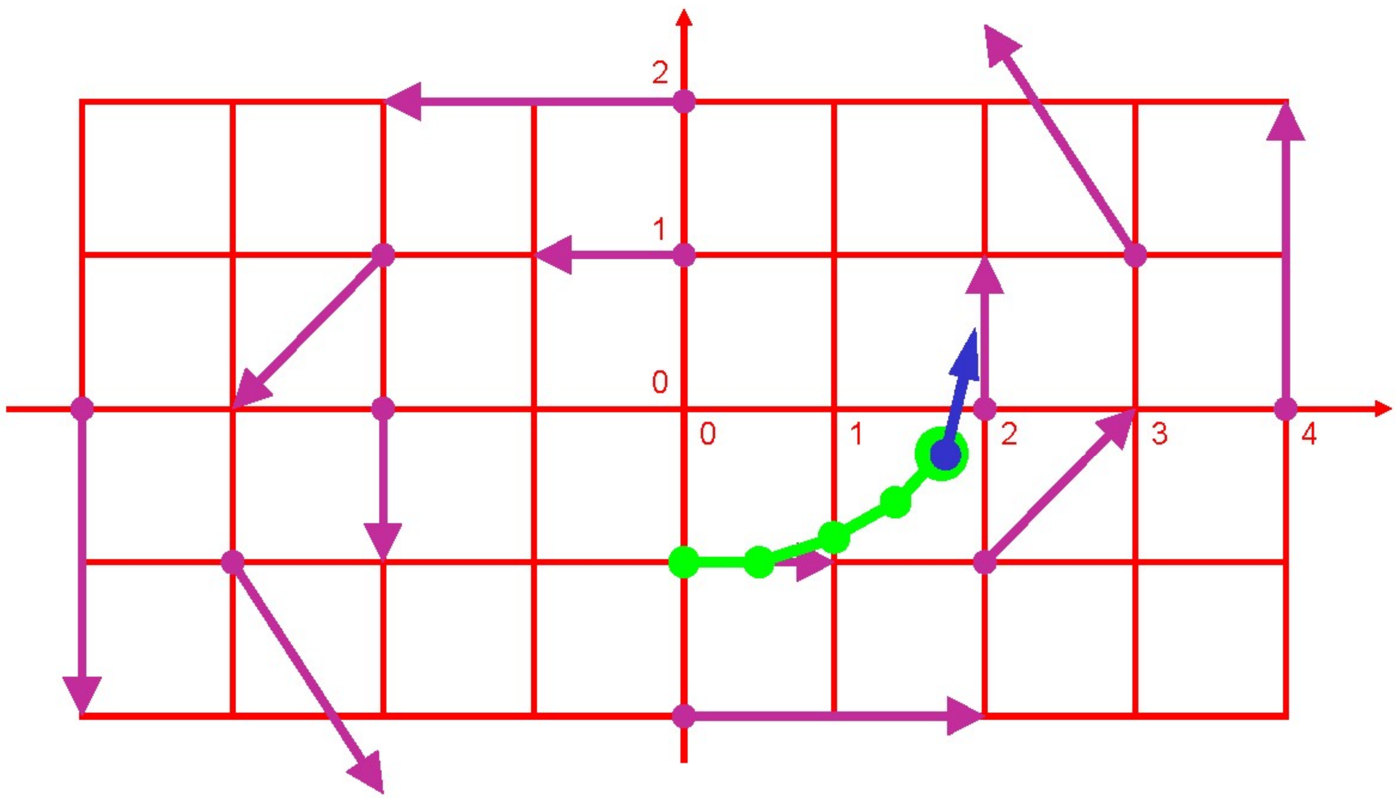
Euler Integration – Example

- $\mathbf{s}_3 = (23/16\,|\,-5/8)^T \approx (1.44\,|\,-0.63)^T;$
  $\mathbf{v}(\mathbf{s}_3) = (5/8\,|\,23/32)^T \approx (0.63\,|\,0.72)^T;$



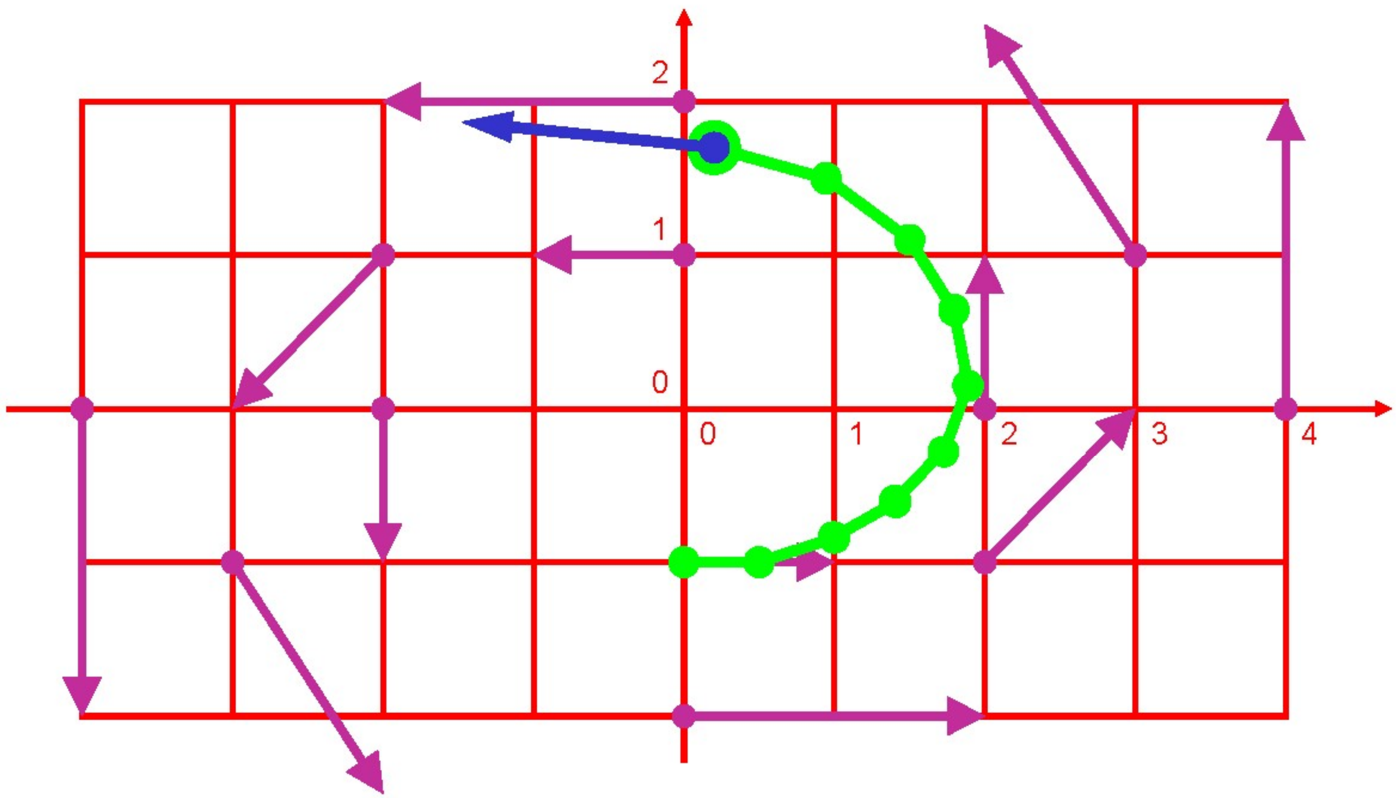
Euler Integration – Example

- $\mathbf{s}_4 = (7/4\,|\,{-17/64})^T \approx (1.75\,|\,{-0.27})^T$;
  $\mathbf{v}(\mathbf{s}_4) = (17/64\,|\,7/8)^T \approx (0.27\,|\,0.88)^T$;



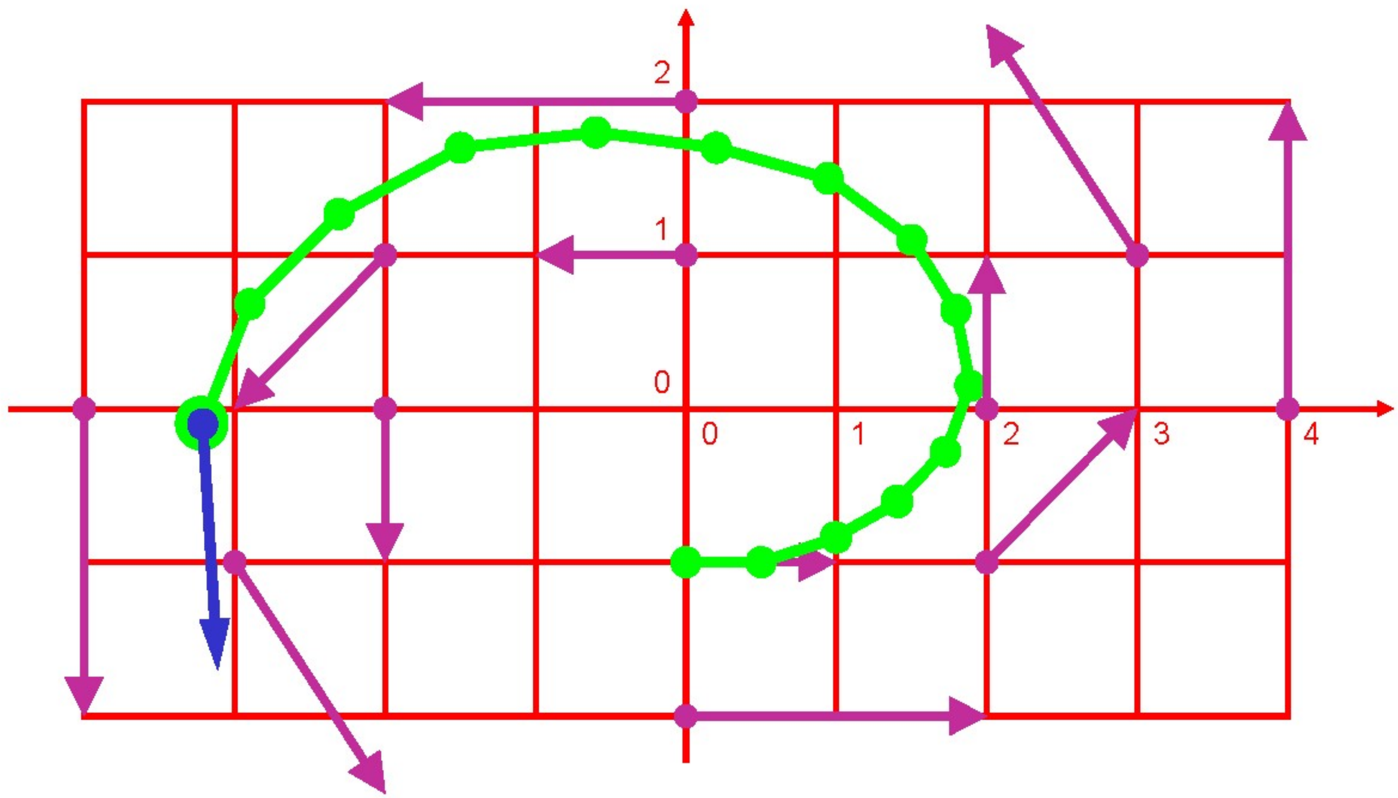
Euler Integration – Example

- $\mathbf{s}_9 \approx (0.20|1.69)^T;$
  $\mathbf{v}(\mathbf{s}_9) \approx (-1.69|0.10)^T;$



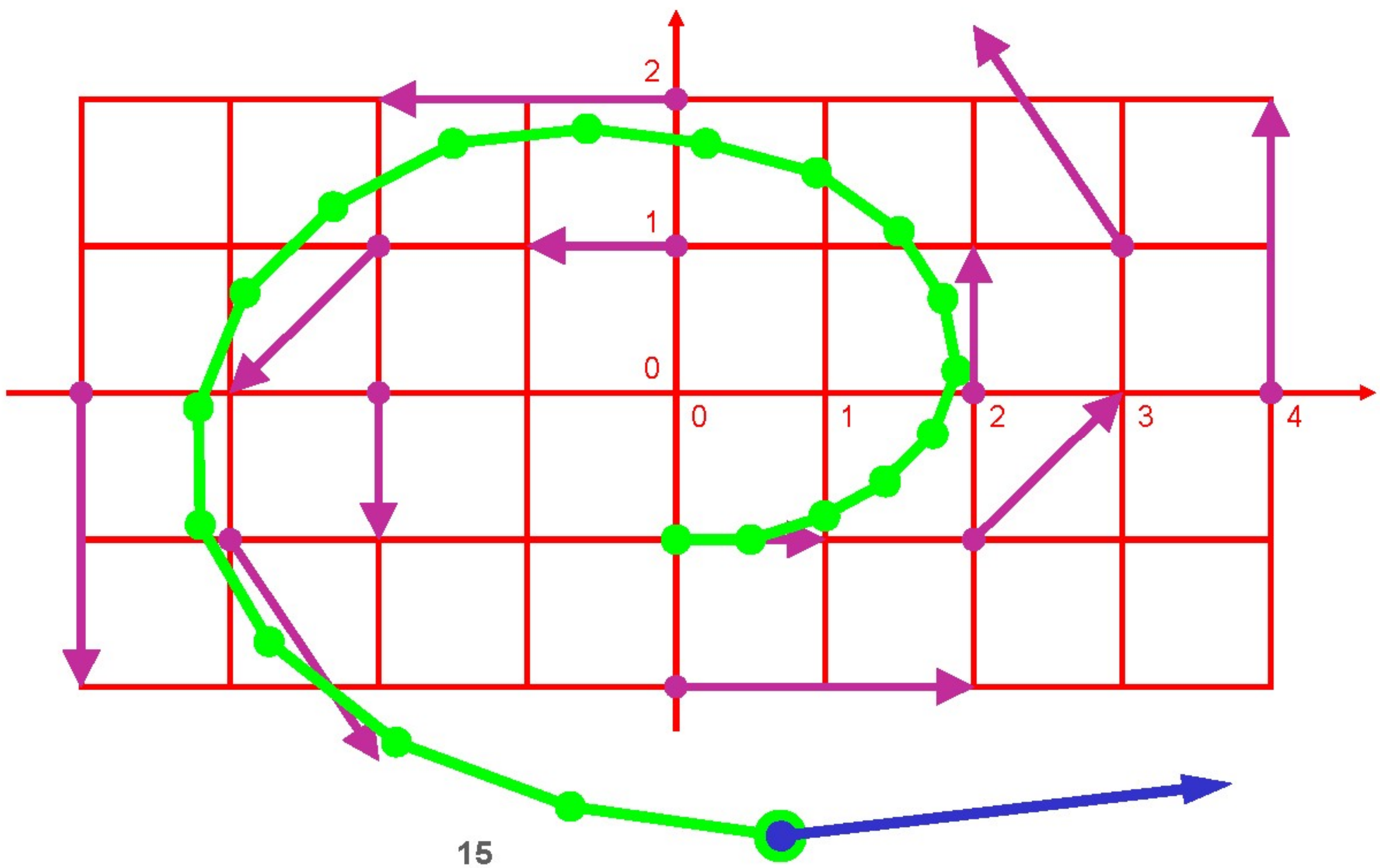
Euler Integration – Example

- $\mathbf{s}_{14} \approx (-3.22 \mid -0.10)^T$;
  $\mathbf{v}(\mathbf{s}_{14}) \approx (0.10 \mid -1.61)^T$;



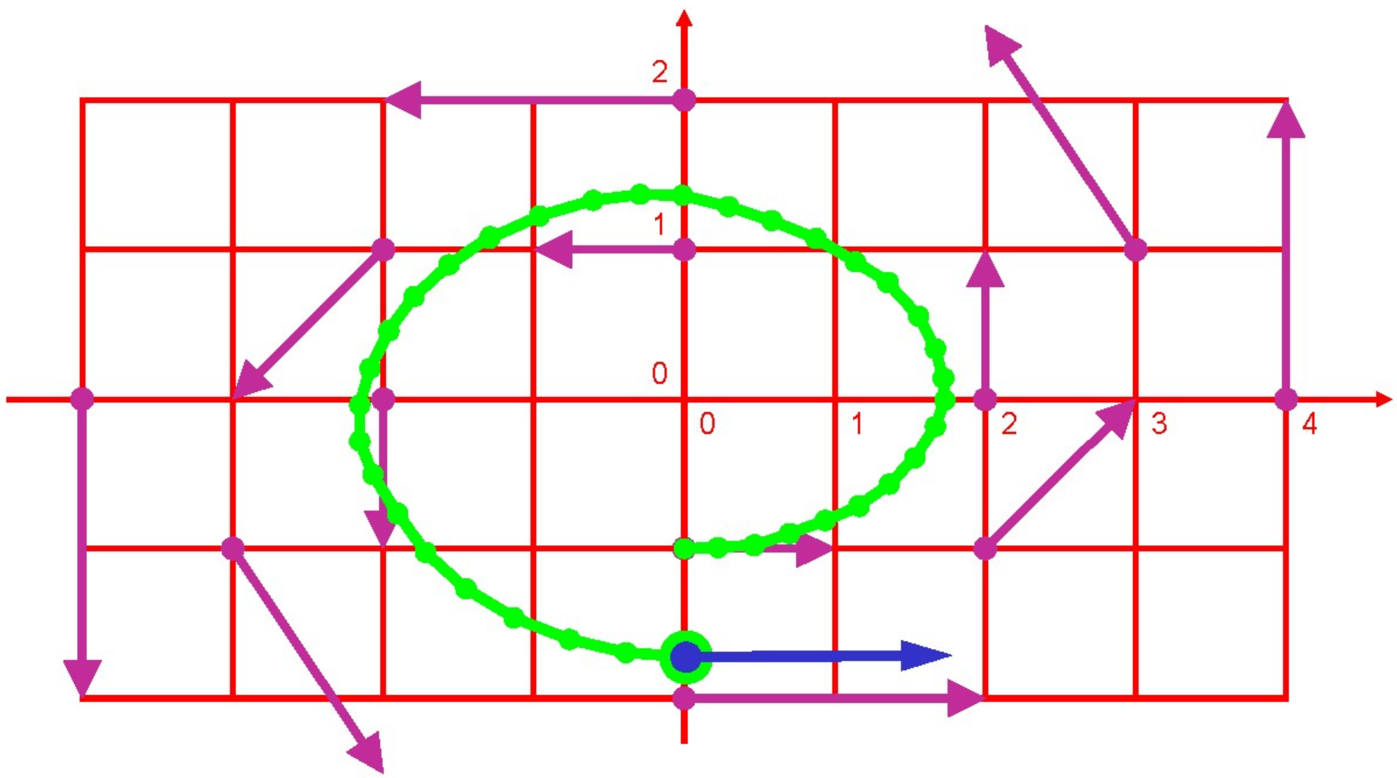
Euler Integration – Example

- $\mathbf{s}_{19} \approx (0.75|\text{-}3.02)^T$; $\mathbf{v}(\mathbf{s}_{19}) \approx (3.02|0.37)^T$;
  clearly: large integration error, dt too large!
  19 steps



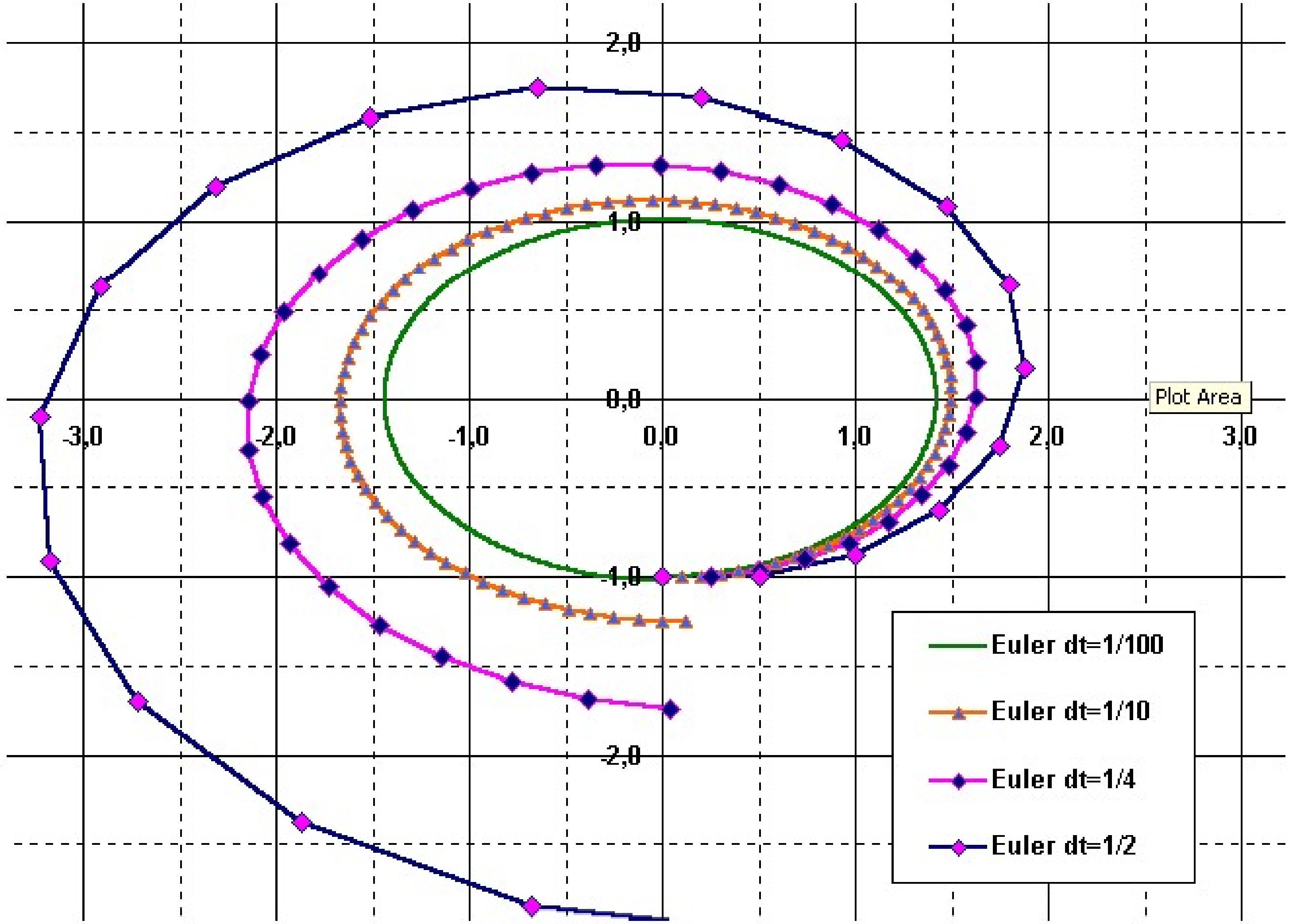
Euler Integration – Example

- dt smaller (1/4): more steps, more exact!
  $\mathbf{s}_{36} \approx (0.04\,|\,-1.74)^T$; $\mathbf{v}(\mathbf{s}_{36}) \approx (1.74\,|\,0.02)^T$;
- 36 steps



Comparison Euler, Step Sizes

Euler
is getting
better
propor-
tionally
to d*t*

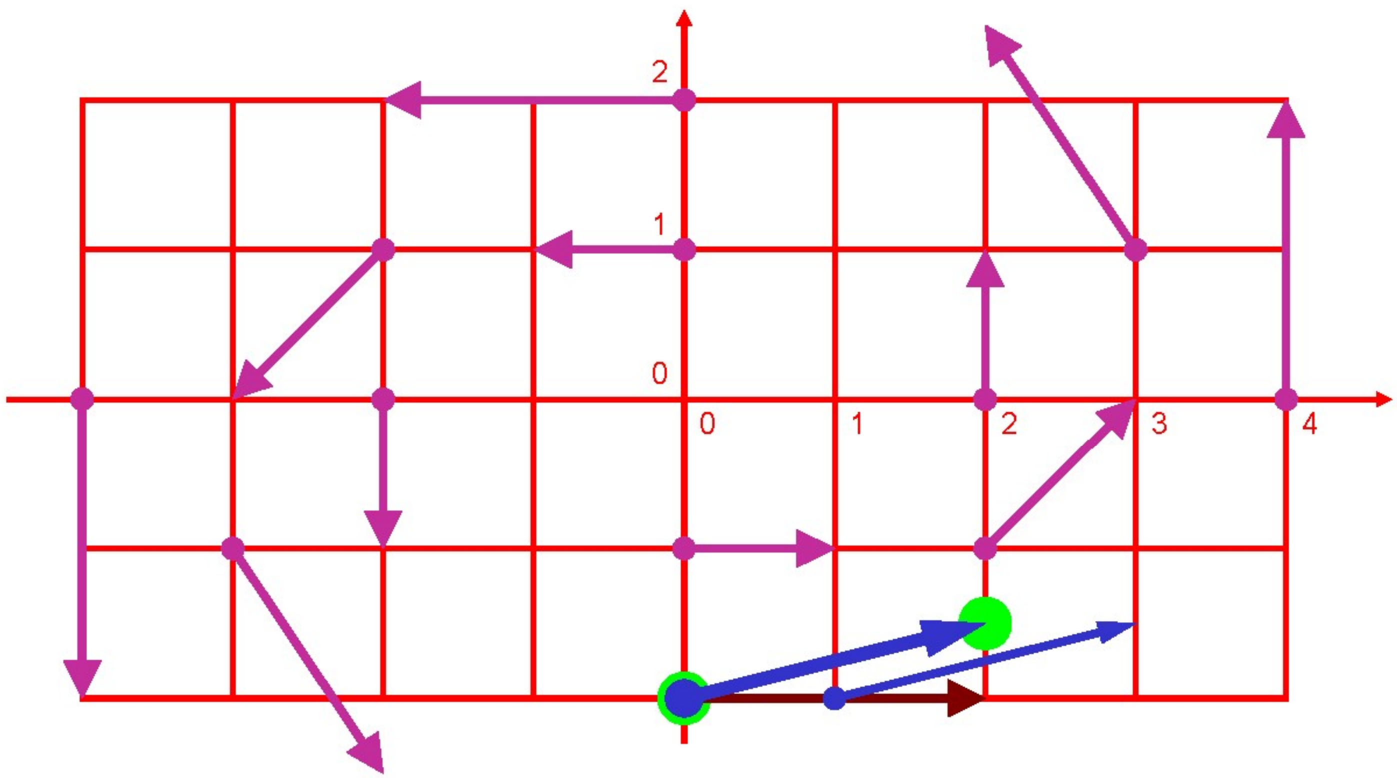


Better than Euler Integr.: RK

- Runge-Kutta Approach:
  - theory: $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{s}(u))\, du$
  - Euler: $\mathbf{s}_i = \mathbf{s}_0 + \sum_{0 \le u < i} \mathbf{v}(\mathbf{s}_u) \cdot dt$
  - Runge-Kutta integration:
    - idea: cut short the curve arc
    - RK-2 (second order RK):
      1.: do half a Euler step
      2.: evaluate flow vector there
      3.: use it in the origin
    - RK-2 (two evaluations of **v** per step):
      $\mathbf{s}_{i+1} = \mathbf{s}_i + \mathbf{v}(\mathbf{s}_i + \mathbf{v}(\mathbf{s}_i) \cdot dt/2) \cdot dt$

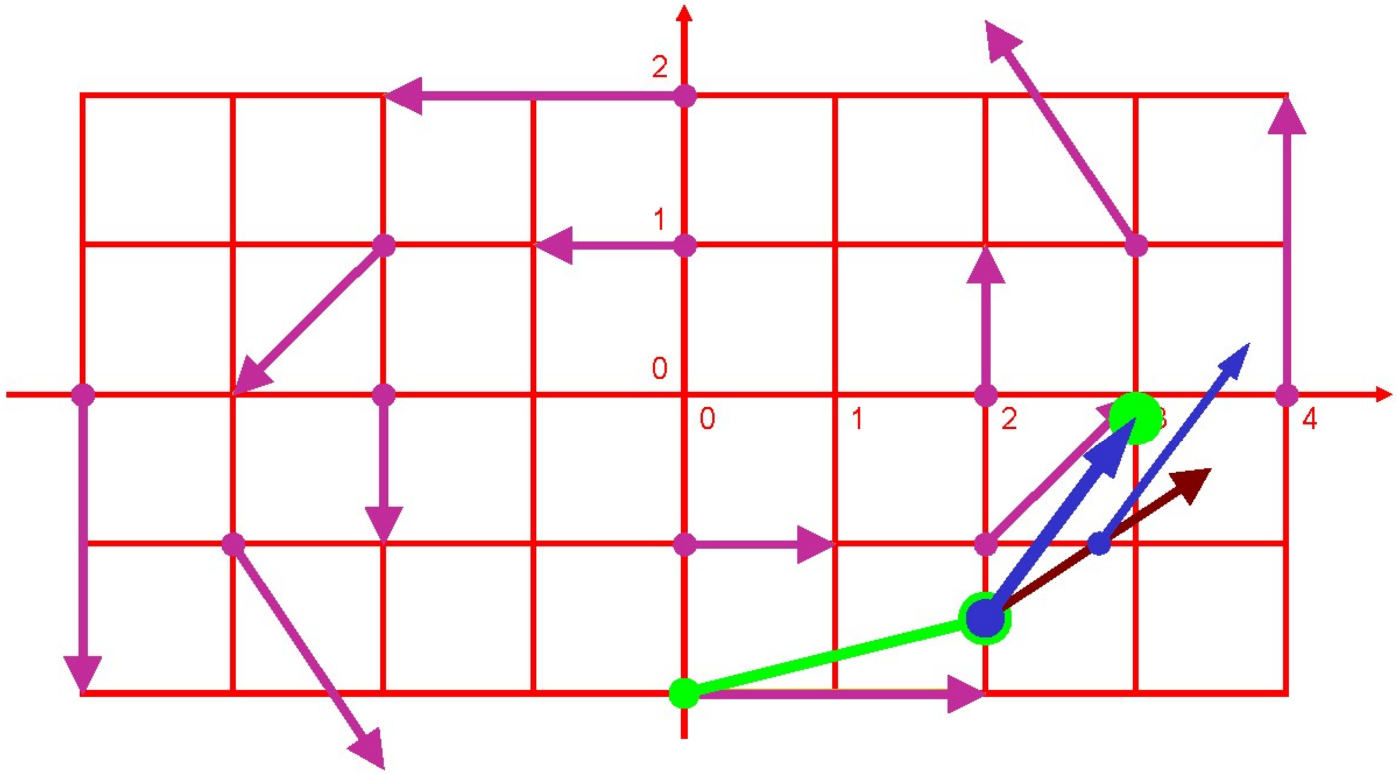
RK-2 Integration – One Step

- Seed point $\mathbf{s}_0 = (0|-2)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_0) = (2|0)^T$;
  preview vector $\mathbf{v}(\mathbf{s}_0+\mathbf{v}(\mathbf{s}_0)\cdot dt/2) = (2|0.5)^T$;
  $dt = 1$



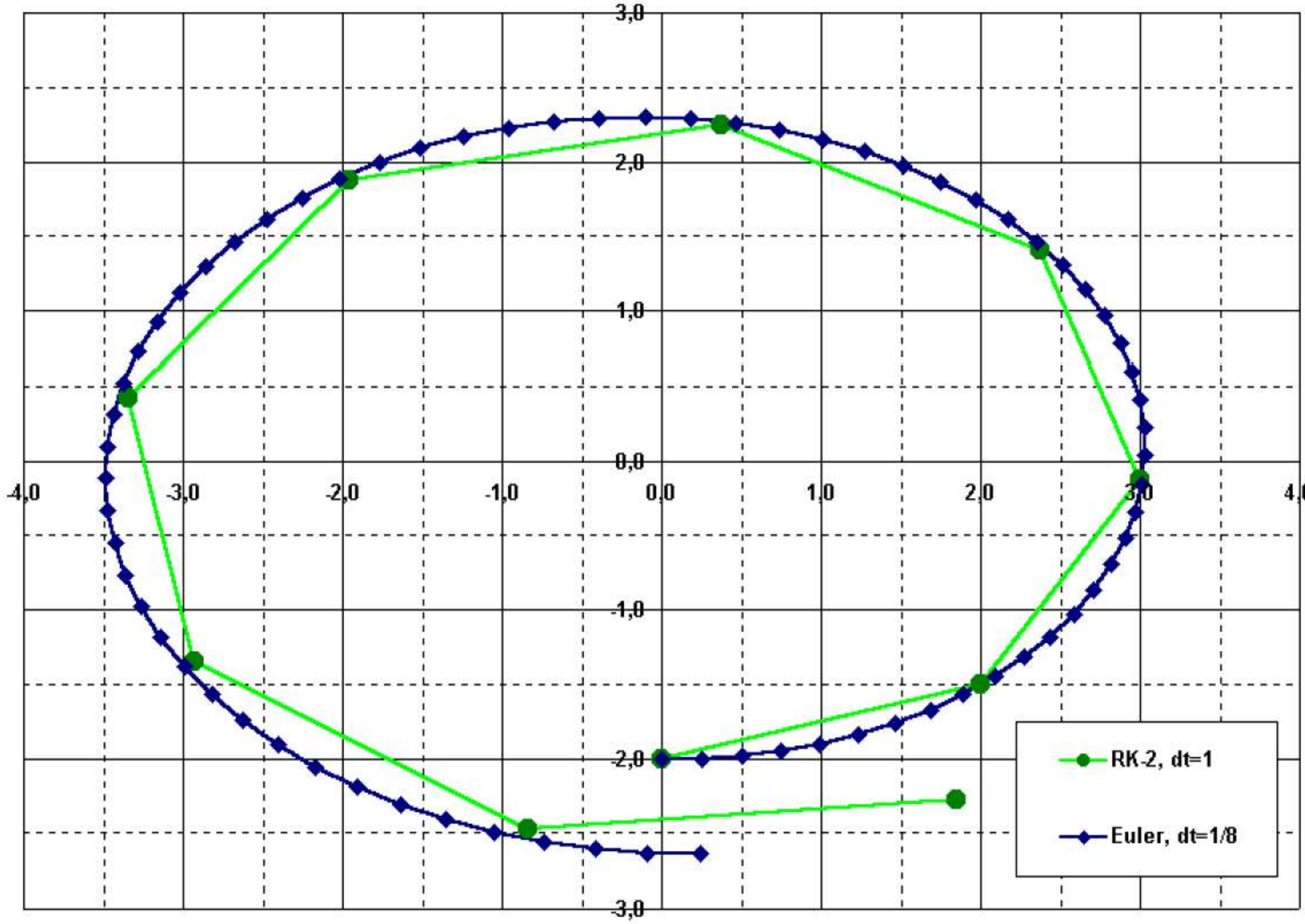
RK-2 – One more step

- Seed point $\mathbf{s}_1 = (2|-1.5)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_1) = (1.5|1)^T$;
  preview vector $\mathbf{v}(\mathbf{s}_1+\mathbf{v}(\mathbf{s}_1)\cdot dt/2) \approx (1|1.4)^T$;
  $dt = 1$



RK-2 – A Quick Round

- RK-2: even with dt=1 (9 steps) better than Euler with dt=1/8 (72 steps)

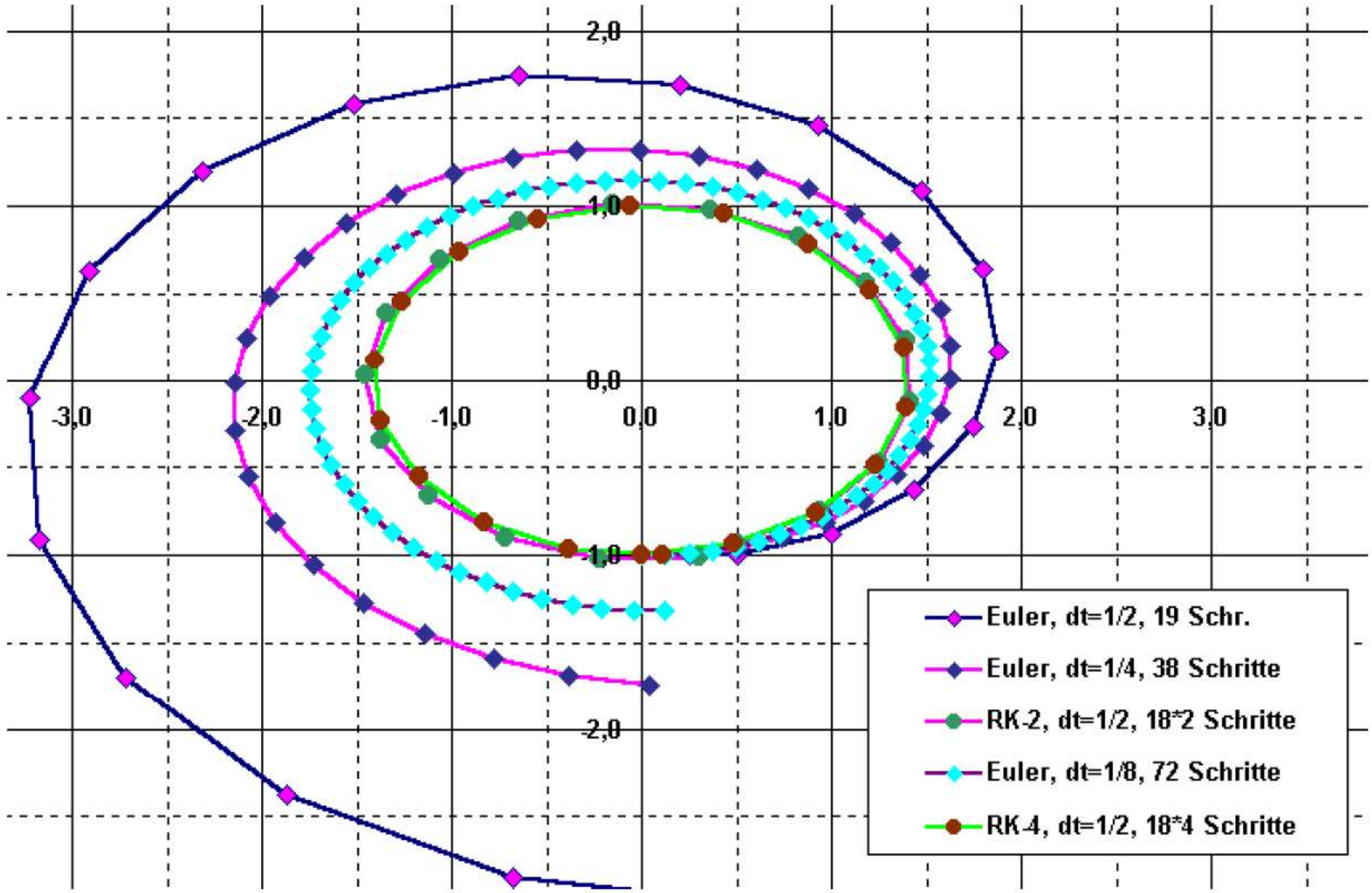


RK-4 vs. Euler, RK-2

- Even better: fourth order RK:
  - four vectors **a**, **b**, **c**, **d**
  - one step is a convex combination:
    $\mathbf{s}_{i+1} = \mathbf{s}_i + (\mathbf{a} + 2 \cdot \mathbf{b} + 2 \cdot \mathbf{c} + \mathbf{d})/6$
  - vectors:
    - $\mathbf{a} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i)$ … original vector
    - $\mathbf{b} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{a}/2)$ … RK-2 vector
    - $\mathbf{c} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{b}/2)$ … use RK-2 …
    - $\mathbf{d} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{c})$ … and again!

Euler vs. Runge-Kutta

- RK-4: pays off only with complex flows
- Here approx. like RK-2



Integration, Conclusions

- Summary:
  - analytic determination of streamlines usually not possible
  - hence: numerical integration
  - several methods available (Euler, Runge-Kutta, etc.)
  - Euler: simple, imprecise, esp. with small dt
  - RK: more accurate in higher orders
  - furthermore: adaptive methods, implicit methods, etc.

Thank you.

Thanks for material

- Helwig Hauser
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