King Abdullah University of Science and Technology KAUST

CS 247 – Scientific Visualization
Lecture 27: Vector / Flow Visualization, Pt. 6

Markus Hadwiger, KAUST

Reading Assignment #14++ (1)

Reading suggestions:

- Data Visualization book, Chapter 6.7
- J. van Wijk: *Image-Based Flow Visualization*,
  ACM SIGGRAPH 2002

  http://www.win.tue.nl/~vanwijk/ibfv/ibfv.pdf

- T. Günther, A. Horvath, W. Bresky, J. Daniels, S. A. Buehler:
  Lagrangian Coherent Structures and Vortex Formation in High Spatiotemporal-Resolution Satellite Winds of an Atmospheric Karman Vortex Street, 2021

  https://www.essoar.org/doi/10.1002/essoar.10506682.2

- H. Bhatia, G. Norgard, V. Pascucci, P.-T. Bremer:
  The Helmholtz-Hodge Decomposition – A Survey, TVCG 19(8), 2013

  https://doi.org/10.1109/TVCG.2012.316

- Work through online tutorials of multi-variable partial derivatives, grad, div, curl, Laplacian:

  https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives
  https://www.youtube.com/watch?v=rB83DpBJQsE (3Blue1Brown)

- Matrix exponentials:

  https://www.youtube.com/watch?v=O85OWBJ2ayo (3Blue1Brown)

Reading Assignment #14++ (2)

Reading suggestions:

- Tobias Günther, Irene Baeza Rojo:

  Introduction to Vector Field Topology

  `https://cgl.ethz.ch/Downloads/Publications/Papers/2020/Gun20b/Gun20b.pdf`

- Roxana Bujack, Lin Yan, Ingrid Hotz, Christoph Garth, Bei Wang:

  State of the Art in Time-Dependent Flow Topology: Interpreting Physical Meaningfulness Through Mathematical Properties

  `https://onlinelibrary.wiley.com/doi/epdf/10.1111/cgf.14037`

- B. Jobard, G. Erlebacher, M. Y. Hussaini:

  Lagrangian-Eulerian Advection of Noise and Dye Textures for Unsteady Flow Visualization

  `http://dx.doi.org/10.1109/TVCG.2002.1021575`

- Anna Vilanova, S. Zhang, Gordon Kindlmann, David Laidlaw:

  An Introduction to Visualization of Diffusion Tensor Imaging and Its Applications

  `http://vis.cs.brown.edu/docs/pdf/Vilanova-2005-IVD.pdf`

Quiz #3: May 10

Organization

- First 30 min of lecture
- No material (book, notes, ...) allowed

Content of questions

- Lectures (both actual lectures and slides)
- Reading assignments (except optional ones)
- Programming assignments (algorithms, methods)
- Solve short practical examples

Vector Fields and Dynamical Systems (1)

Velocity gradient tensor, (vector field → tensor field)

- Gradient of vector field: how does the vector field change?
- In Cartesian coordinates: *spatial partial derivatives (Jacobian matrix)*

$$\nabla \mathbf{v}\,(x,y,z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

these are partial derivatives!

- Can be decomposed into *symmetric* part + *anti-symmetric* part

$\nabla \mathbf{v} = \mathbf{D} + \mathbf{S}$ *velocity gradient tensor*

sym.: $\mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^{T} \right)$ deform.: *rate-of-strain tensor*

skew-sym.: $\mathbf{S} = \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^{T} \right)$ rotation: *vorticity/spin tensor*

Vector Fields and Dynamical Systems (2)

Vorticity/spin/angular velocity tensor

- Antisymmetric part of velocity gradient tensor
- Corresponds to vorticity/curl/angular velocity (beware of factor ½)

$$\mathbf{S} = \tfrac{1}{2} \left(\nabla \mathbf{v} - (\nabla \mathbf{v})^{T} \right)$$

these are partial derivatives!

$$\mathbf{S} = \tfrac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

$\mathbf{S}$ acts on vector like cross product with ω : $\mathbf{S} \bullet = \tfrac{1}{2}\, \omega \times$

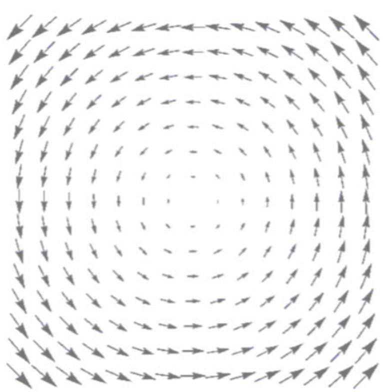
$$\mathbf{v}^{(r)} = \mathbf{S} \cdot d\mathbf{r} = \frac{1}{2} \left[\nabla \mathbf{v} - (\nabla \mathbf{v})^{T} \right] \cdot d\mathbf{r} = \frac{1}{2} \boldsymbol{\omega} \times d\mathbf{r}$$

Angular Velocity of Rigid Body Rotation

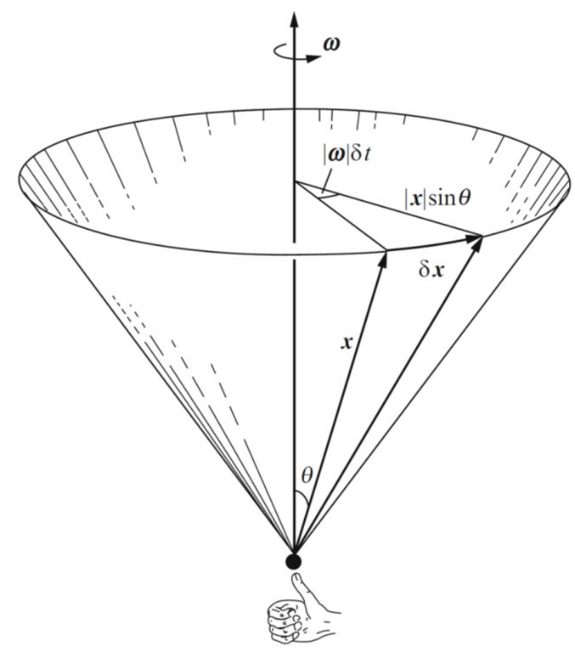
Rate of rotation

- Scalar ω: angular displacement per unit time (rad s^{-1})
  - Angle Θ at time t is $\Theta(t) = \omega t$; $\omega = 2\pi f$ where f is the frequency ($f = 1/T$; s^{-1})
- Vector $\boldsymbol{\omega}$: axis of rotation; magnitude is angular speed (if $\boldsymbol{\omega}$ is curl: speed x2)
  - Beware of different conventions that differ by a factor of ½ !

Cross product of ½$\boldsymbol{\omega}$ with vector to center of rotation ($\mathbf{r}$) gives linear velocity vector $\mathbf{v}$ (tangent)



$$\mathbf{v}^{(r)} = \frac{1}{2}\boldsymbol{\omega} \times d\mathbf{r}$$



Velocity Gradient Tensor and Components (1)

Velocity gradient tensor

(here: in Cartesian coordinates)

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial}{\partial x} v^x & \frac{\partial}{\partial y} v^x & \frac{\partial}{\partial z} v^x \\ \frac{\partial}{\partial x} v^y & \frac{\partial}{\partial y} v^y & \frac{\partial}{\partial z} v^y \\ \frac{\partial}{\partial x} v^z & \frac{\partial}{\partial y} v^z & \frac{\partial}{\partial z} v^z \end{bmatrix}$$

these are the same partial derivatives as before!

$$\nabla \mathbf{v} = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right) + \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^T\right)$$

Velocity Gradient Tensor and Components (2)

Rate-of-strain (rate-of-deformation) tensor

(symmetric part; here: in Cartesian coordinates)

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} 2\frac{\partial}{\partial x}v^x & \frac{\partial}{\partial y}v^x + \frac{\partial}{\partial x}v^y & \frac{\partial}{\partial z}v^x + \frac{\partial}{\partial x}v^z \\ \frac{\partial}{\partial x}v^y + \frac{\partial}{\partial y}v^x & 2\frac{\partial}{\partial y}v^y & \frac{\partial}{\partial z}v^y + \frac{\partial}{\partial y}v^z \\ \frac{\partial}{\partial x}v^z + \frac{\partial}{\partial z}v^x & \frac{\partial}{\partial y}v^z + \frac{\partial}{\partial z}v^y & 2\frac{\partial}{\partial z}v^z \end{bmatrix}$$

$$tr(\mathbf{D}) = \nabla \cdot \mathbf{v}$$

Velocity Gradient Tensor and Components (3)

Vorticity tensor (spin tensor)

(skew-symmetric part; here: in Cartesian coordinates)

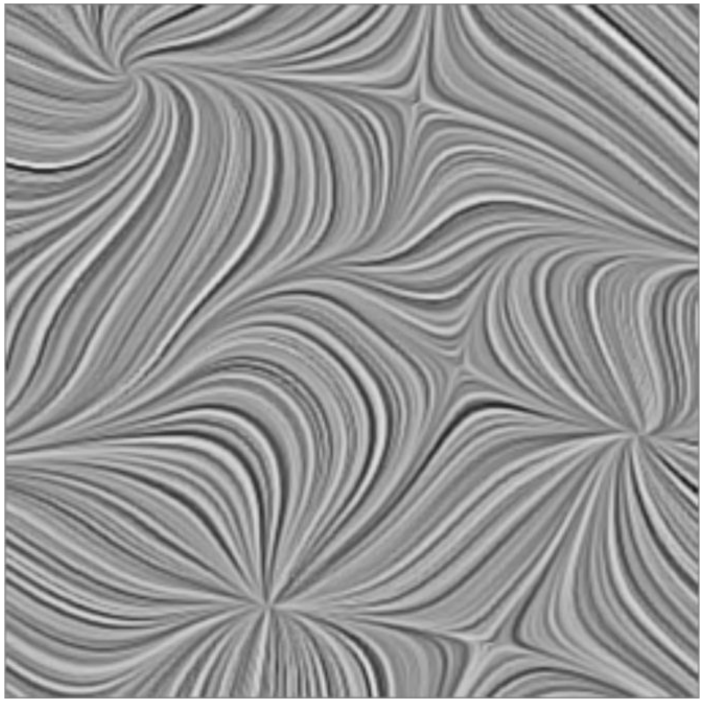
$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial}{\partial y}v^x - \frac{\partial}{\partial x}v^y & \frac{\partial}{\partial z}v^x - \frac{\partial}{\partial x}v^z \\ \frac{\partial}{\partial x}v^y - \frac{\partial}{\partial y}v^x & 0 & \frac{\partial}{\partial z}v^y - \frac{\partial}{\partial y}v^z \\ \frac{\partial}{\partial x}v^z - \frac{\partial}{\partial z}v^x & \frac{\partial}{\partial y}v^z - \frac{\partial}{\partial z}v^y & 0 \end{bmatrix}$$

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \qquad \boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$$

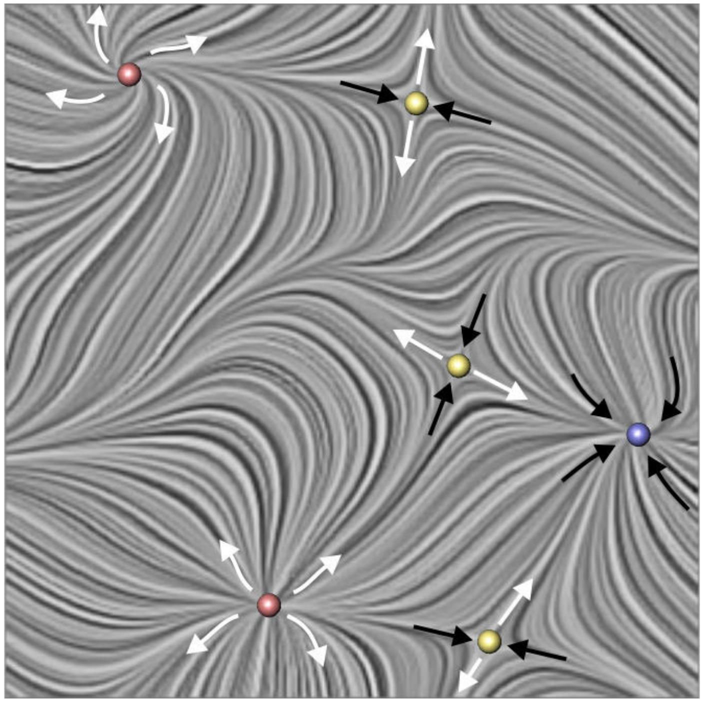
Critical Point Analysis

Critical Points (Steady Flow!)

Classify critical points according to the *eigenvalues* of the velocity gradient tensor at the critical point



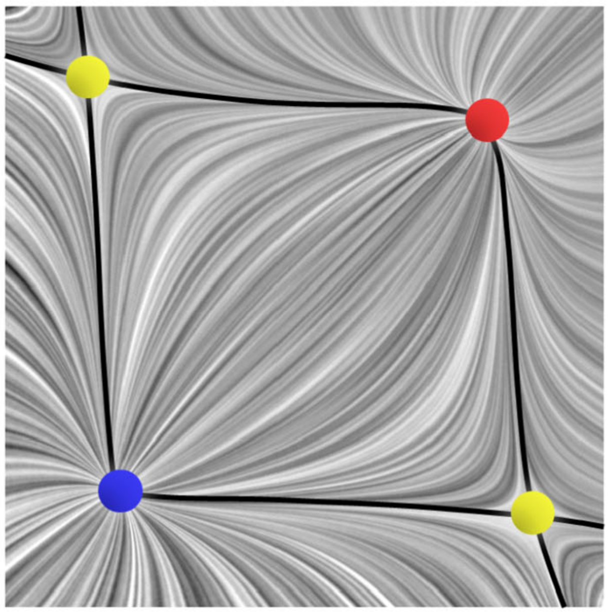
stream lines (LIC)

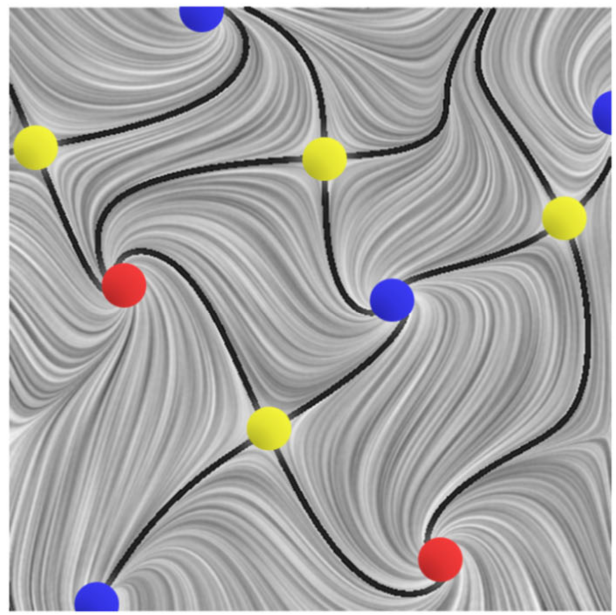


critical points (**v** = 0)

Vector Field Topology: Topological Skeleton

Connect critical points by *separatrices*





Sources (red), sinks (blue), saddles (yellow)

(Non-Linear) Dynamical Systems

Start with system of linear ODEs (with constant coefficients)

- Non-linear systems can be linearized around critical points
- Use linearization for characterization

$\dot{\mathbf{x}} = A\mathbf{x}$ $\quad$ A is an $n \times n$ matrix $\quad \rightarrow$

$$\mathbf{v} = A\mathbf{x}, \qquad \nabla \mathbf{v} = A.$$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

solution: $\mathbf{x}(t) = e^{At}\mathbf{x}_0$

characterize behavior
through eigenvalues of A

A Few Facts about Eigenvalues and –vectors

The matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ has eigenvalues $\lambda_1 = c + s\mathbf{i} \quad \lambda_2 = c - s\mathbf{i}$

with eigenvectors $u_1 = \begin{bmatrix} 1 \\ -\mathbf{i} \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ +\mathbf{i} \end{bmatrix}$ (if s non-zero)

If c = 0, this is a skew-symmetric matrix: pure imaginary eigenvalues

Skew-symmetric matrices: "infinitesimal rotations" (infinitesimal generators of rot.)

For $c = \cos\theta$ and $s = \sin\theta$: 2x2 rotation matrix with

$$\lambda_1 = e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$$

$$\lambda_2 = e^{-\mathbf{i}\theta} = \cos\theta - \mathbf{i}\sin\theta$$

Eigenvalues

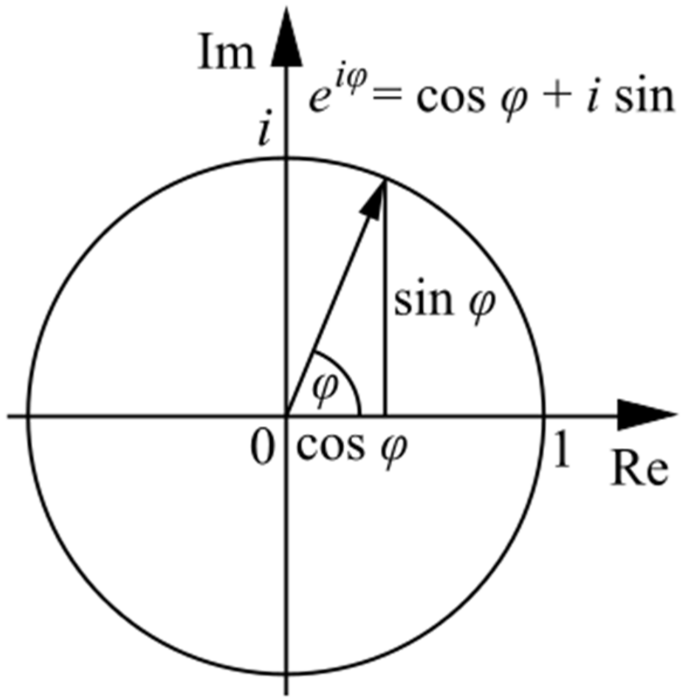
- Symmetric matrix: all eigenvalues are *real*
- Skew-symmetric matrix: all eigenvalues are *pure imaginary*

Euler's Formula

Can be derived from the infinite power series for exp(), cos(), sin()

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi} + 1 = 0$$



Matrix Exponentials

Defined via same power series as usual exponential

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

Easy if *X* is diagonalizable

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Exponentials of anti-symmetric matrices are rotation matrices

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Matrix Exponentials

Defined via same power series as usual exponential

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots$$

Easy if *X* is diagonalizable

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \exp\left(tX\right) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Complex eigenvalues lead to rotation

$$X = \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix} \qquad \exp\left(tX\right) = e^{at} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}$$

$$\lambda_{1,2} = a \pm \mathbf{i}\omega$$

Classification of Critical Points (1)

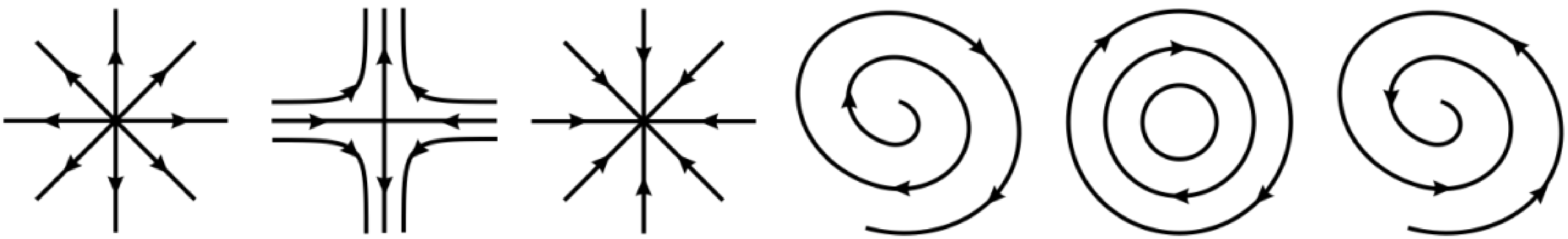
(Isolated) critical point (equilibrium point)

- Velocity vanishes (all components zero)

$$\mathbf{v}(\mathbf{x}_c) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_c \pm \epsilon) \neq \mathbf{0} \qquad \det(\nabla\mathbf{v}(\mathbf{x}_c)) \neq 0$$

Characterize using velocity gradient $\nabla\mathbf{v}$ at critical point $\mathbf{x}_c$

- Look at eigenvalues (and eigenvectors) of $\nabla\mathbf{v}$

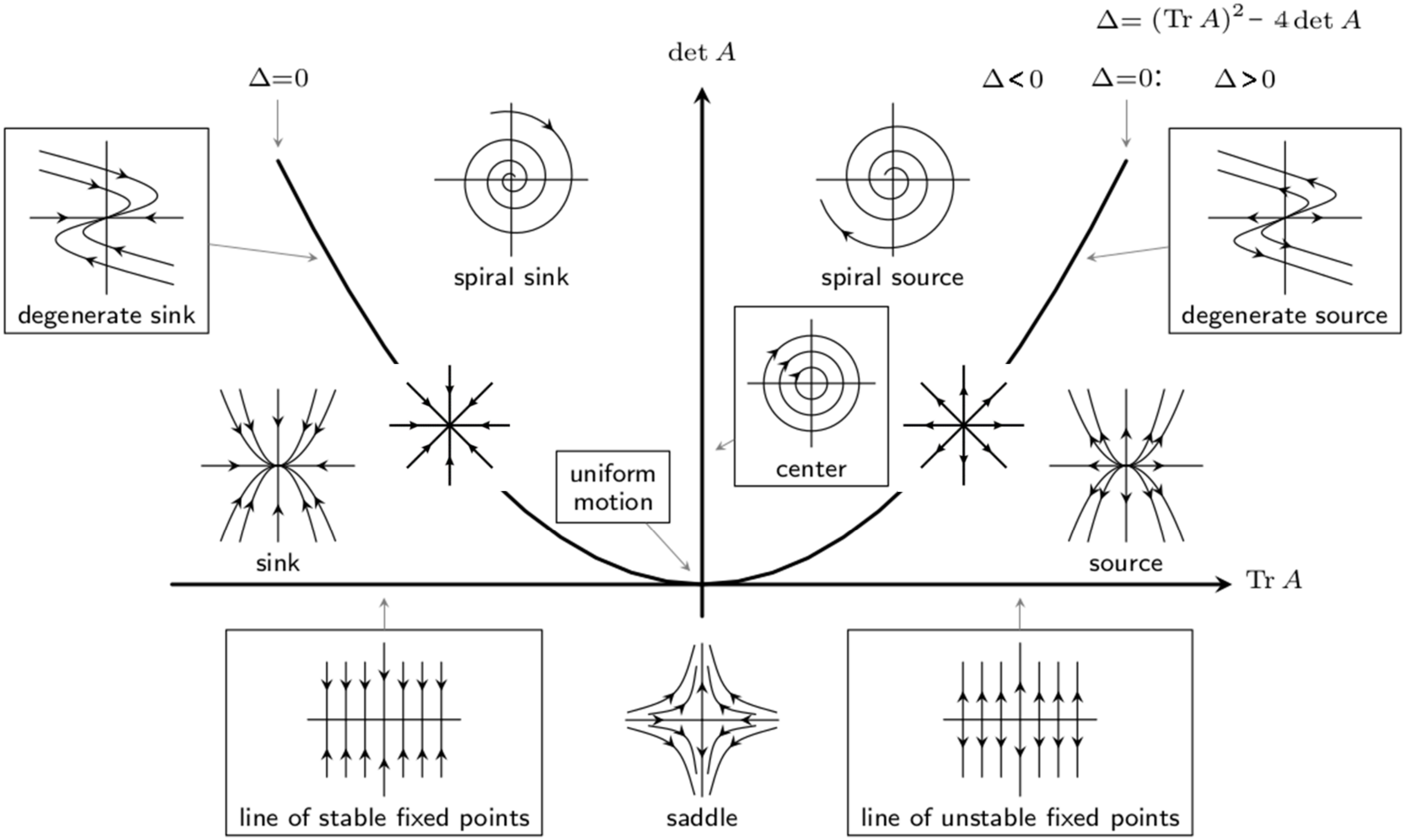


Repelling node	Saddle point	Attracting node	Repelling focus	Center	Attracting focus
$R_1, R_2 > 0$	$R_1 < 0, R_2 > 0$	$R_1, R_2 < 0$	$R_1 = R_2 > 0$	$R_1 = R_2 = 0$	$R_1 = R_2 < 0$
$I_1 = I_2 = 0$	$I_1 = I_2 = 0$	$I_1 = I_2 = 0$	$I_1 = -I_2 \neq 0$	$I_1 = -I_2 \neq 0$	$I_1 = -I_2 \neq 0$

the first three phase portraits are special cases, see later slides!

Classification of Critical Points (2)

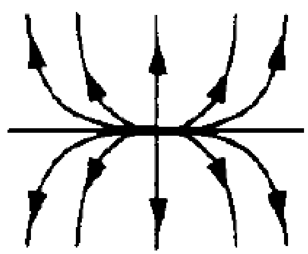
Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \mathrm{Tr}\, A)$-plane



A Few Details (1)

Repelling/attracting nodes

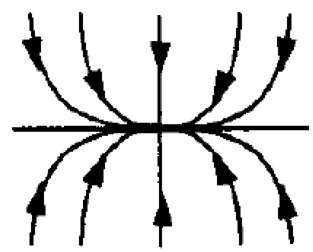
- Do not necessarily imply that streamlines are straight lines
  (do not confuse with the linear system of ODEs!)
- They are only straight lines when both eigenvalues are real and have the same sign, *and are also equal* (as in the phase portraits before)
- If they are not equal:



Repelling Node

R1, R2 > 0

I1, I2 = 0



Attracting Node

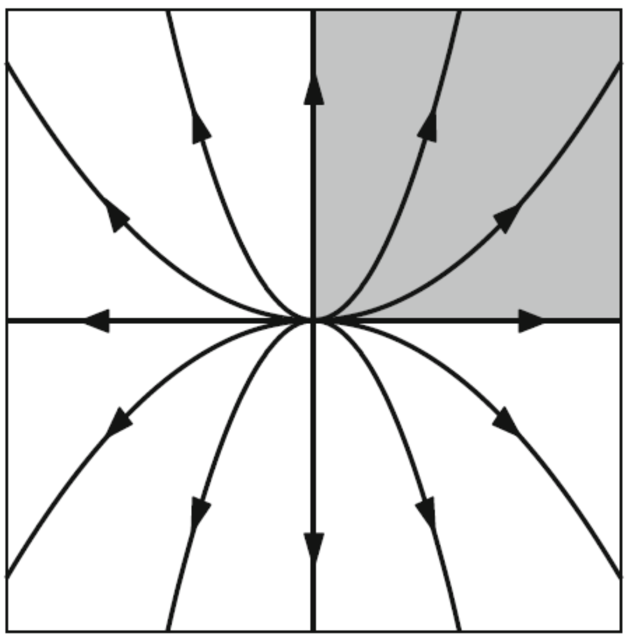
R1, R2 < 0

I1, I2 = 0

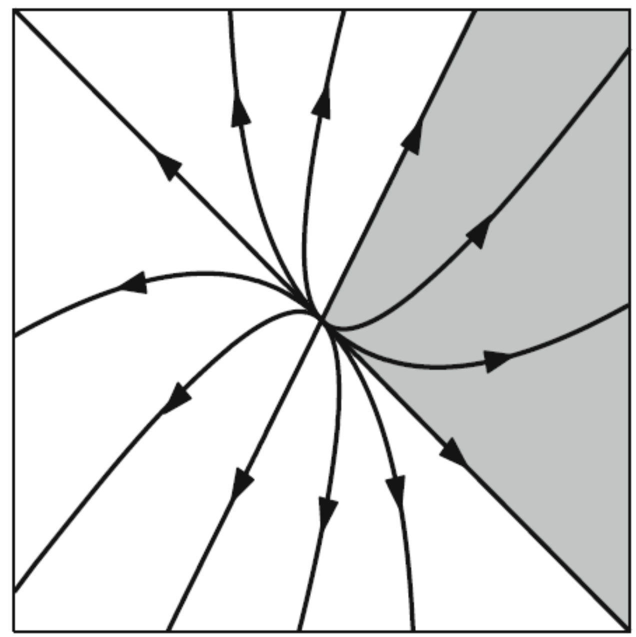
A Few Details (2)

What about skew axes?

- Both of the systems below have eigenvalues 3 and 6
- Jordan normal form (Jordan canonical form) gives details



$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$



$$\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}$$

Jordan Normal Form (2x2 Matrix)

For every real 2x2 matrix A there is an invertible P such that

$P^{-1}AP$ is one of the following Jordan matrices (all entries are real):

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$ (defective matrix)

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Each of these has its corresponding rule for constructing P

- Example on prev. slide (the two eigenvectors are not orthogonal):

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

See also *algebraic* and *geometric multiplicity* of eigenvalues

Jordan Normal Form (2x2 Matrix)

For every real 2x2 matrix A there is an invertible P such that

$P^{-1}AP$ is one of the following Jordan matrices (all entries are real):

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$ (defective matrix)

same eigenvalues, trace, determinant!

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Each of these has its corresponding rule for constructing P

- Example on prev. slide (the two eigenvectors are not orthogonal):

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

See also *algebraic* and *geometric multiplicity* of eigenvalues

Another Example

$P^{-1}AP$ has form J_1

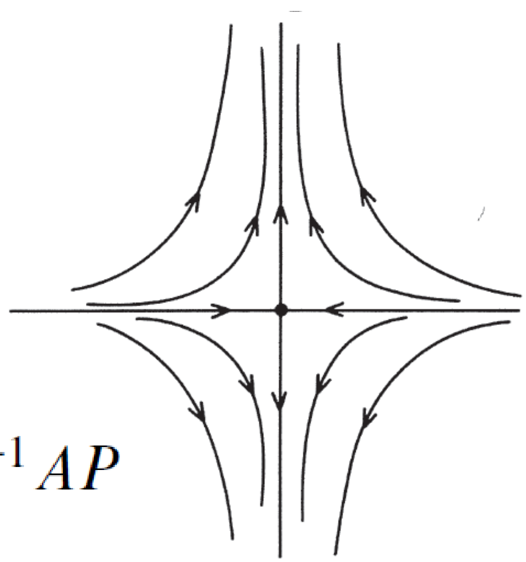
Eigenvalues:

$\lambda_1 = -1$

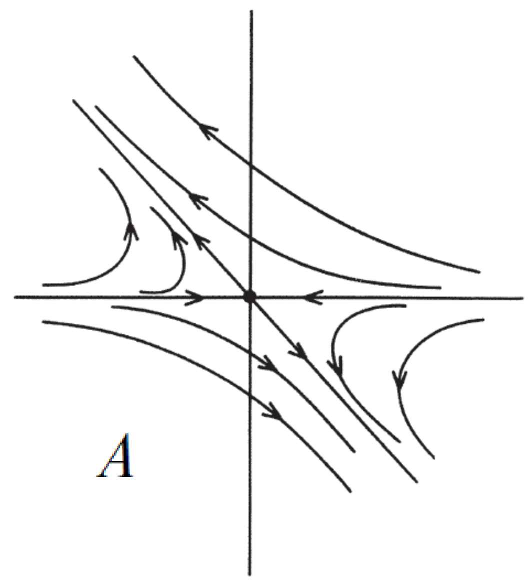
$\lambda_2 = 2$

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



$P^{-1}AP$

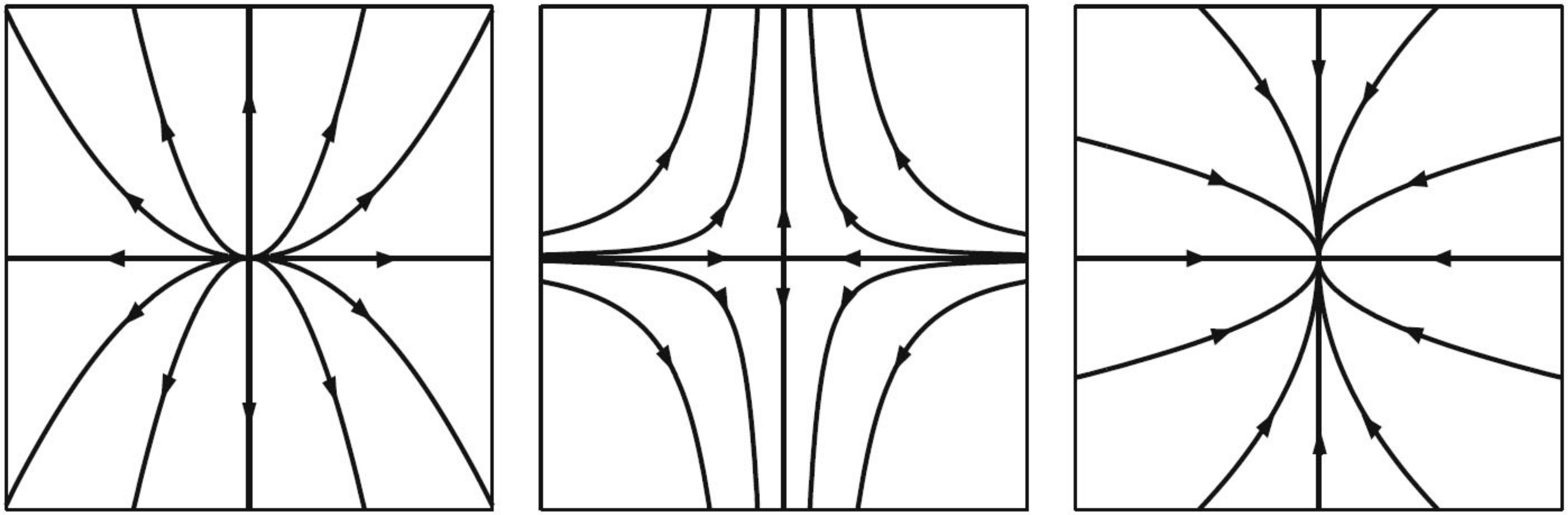


A

Jordan Form Characterization (1)

Phase portraits corresponding to Jordan matrix

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$



$0 < \lambda_1 < \lambda_2$ unstable node

$\lambda_1 < 0 < \lambda_2$ saddle

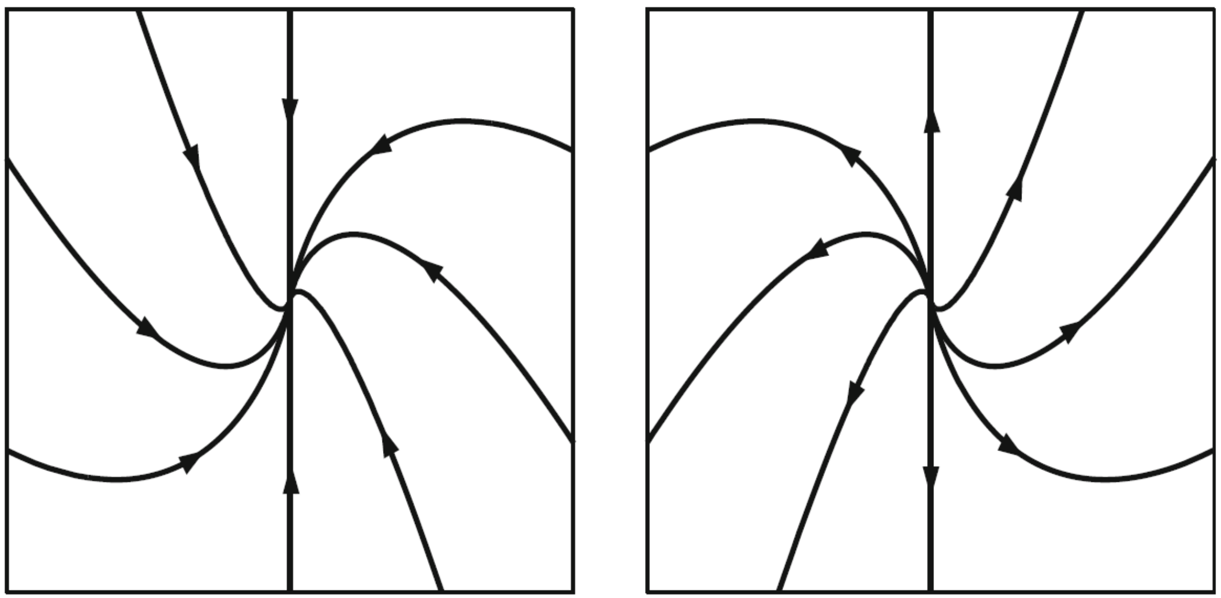
$\lambda_1 < \lambda_2 < 0$ stable node

Jordan Form Characterization (2)

Phase portraits corresponding to Jordan matrix

(matrix is defective: eigenspaces collapse, geometric multiplicity less than algebraic multiplicity)

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$



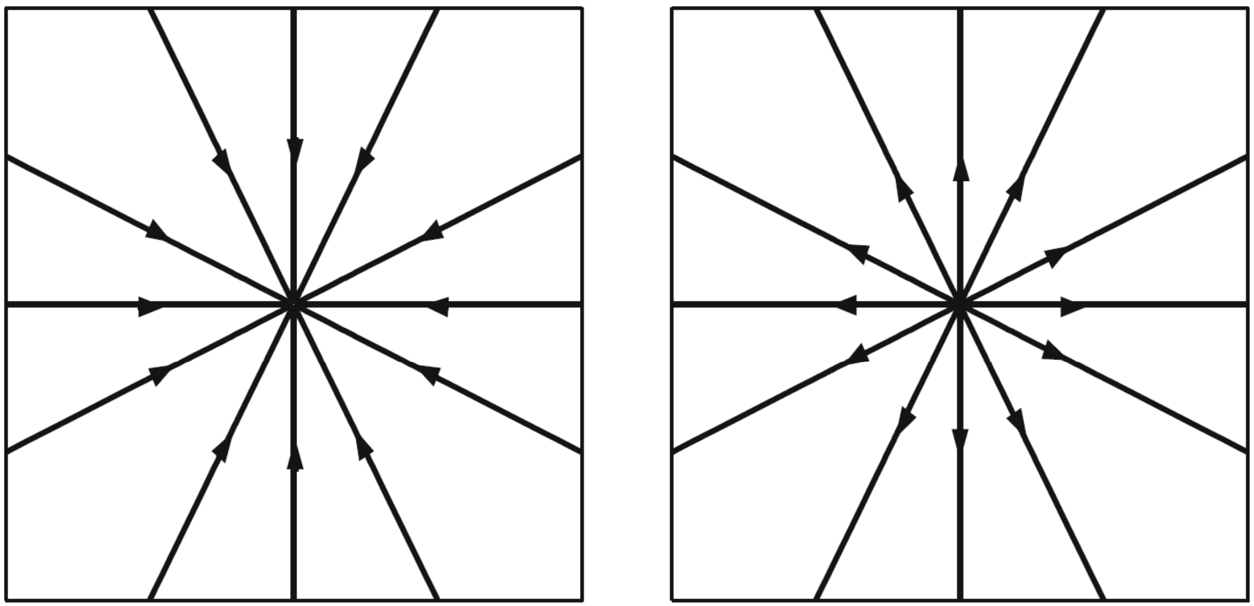
$\lambda < 0$
stable improper node

$\lambda > 0$
unstable improper node

Jordan Form Characterization (3)

Phase portraits corresponding to Jordan matrix

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$



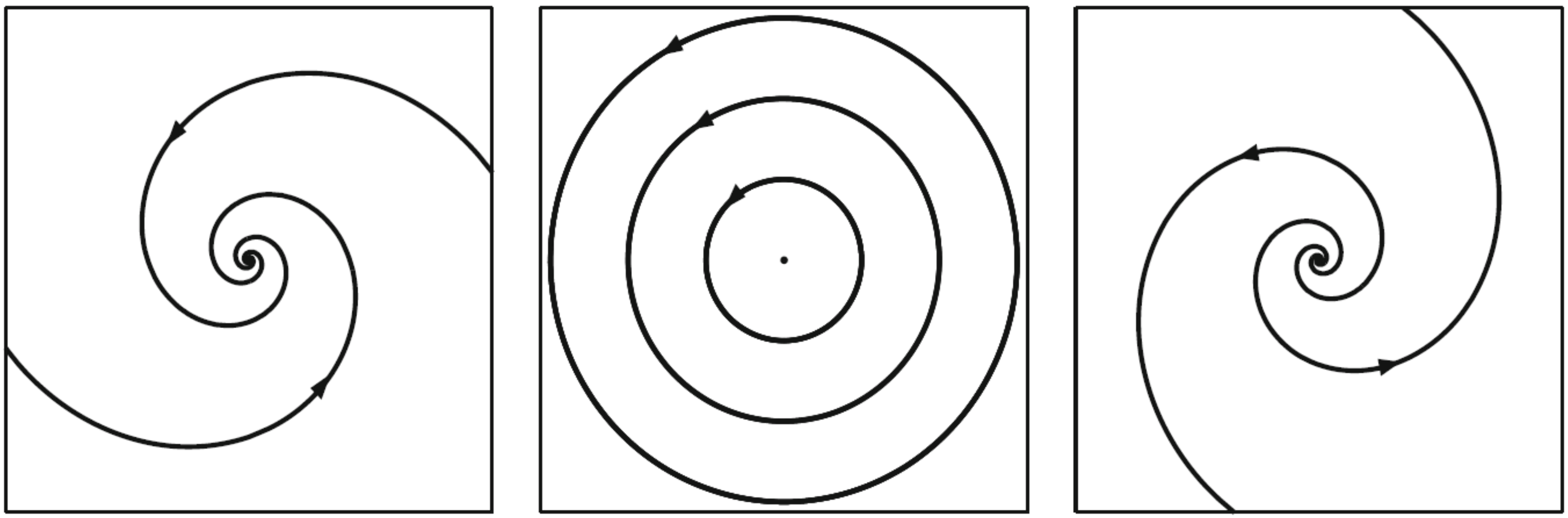
$\lambda < 0$
stable star node

$\lambda > 0$
unstable star node

Jordan Form Characterization (4)

Phase portraits corresponding to Jordan matrix

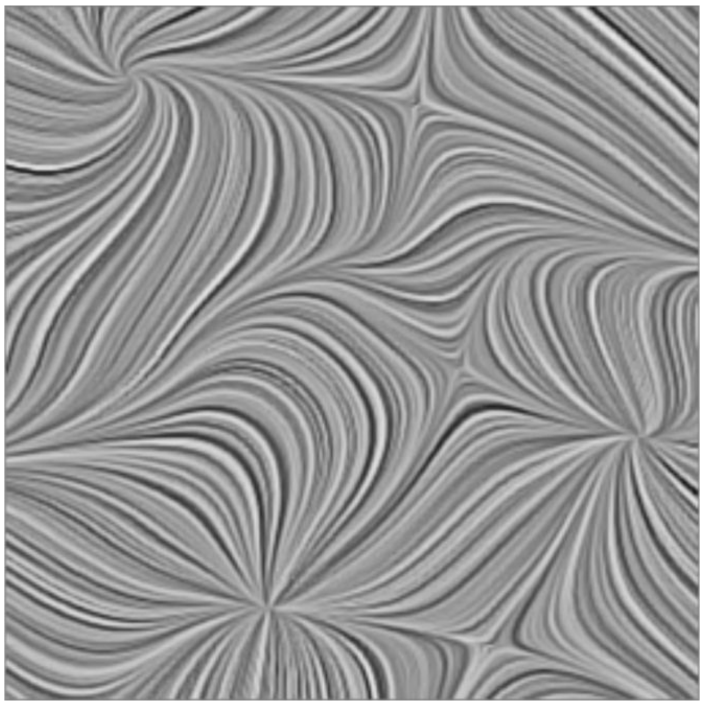
$$J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$



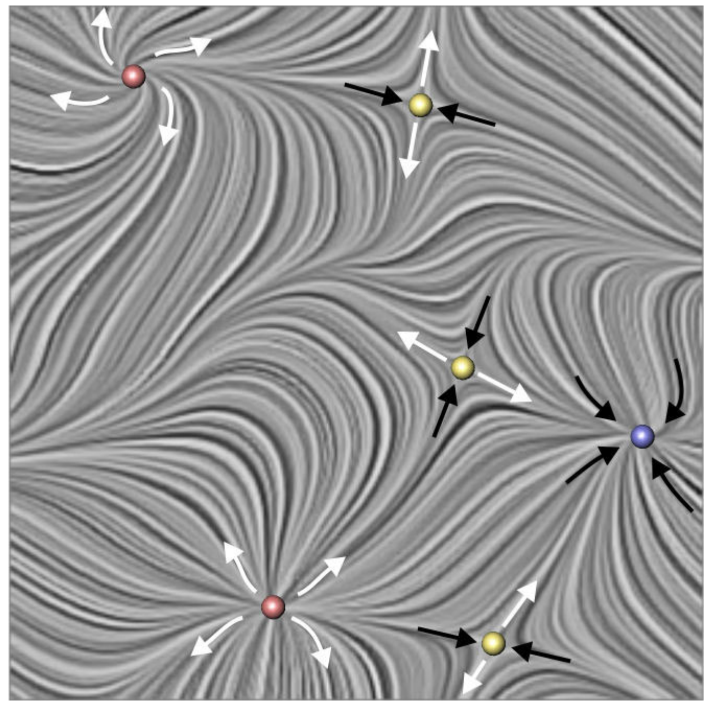
$a < 0$ stable spiral node | $a = 0$ center | $a > 0$ unstable spiral node

Critical Points (Steady Flow!)

Classify critical points according to the *eigenvalues* of the velocity gradient tensor at the critical point



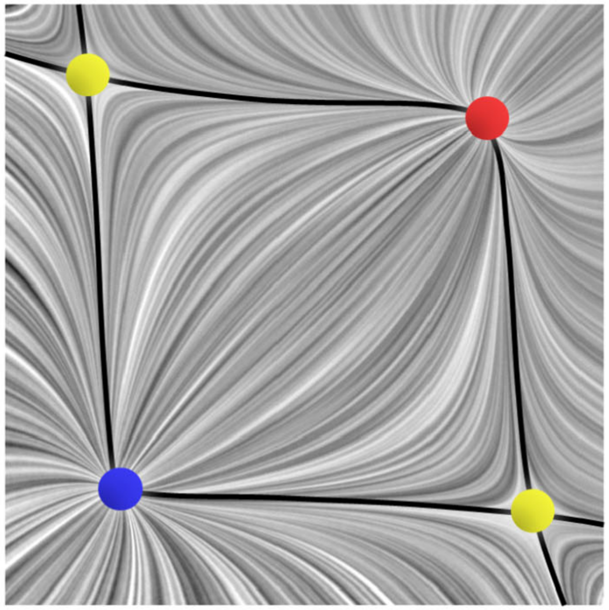
stream lines (LIC)

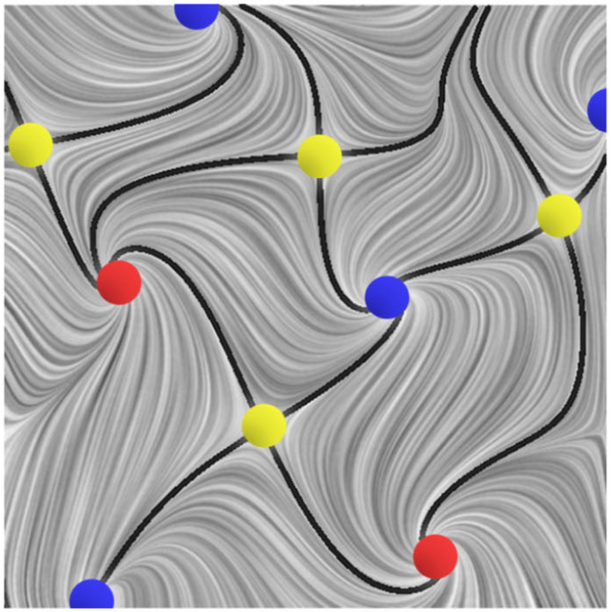


critical points (**v** = 0)

Vector Field Topology: Topological Skeleton

Connect critical points by *separatrices*

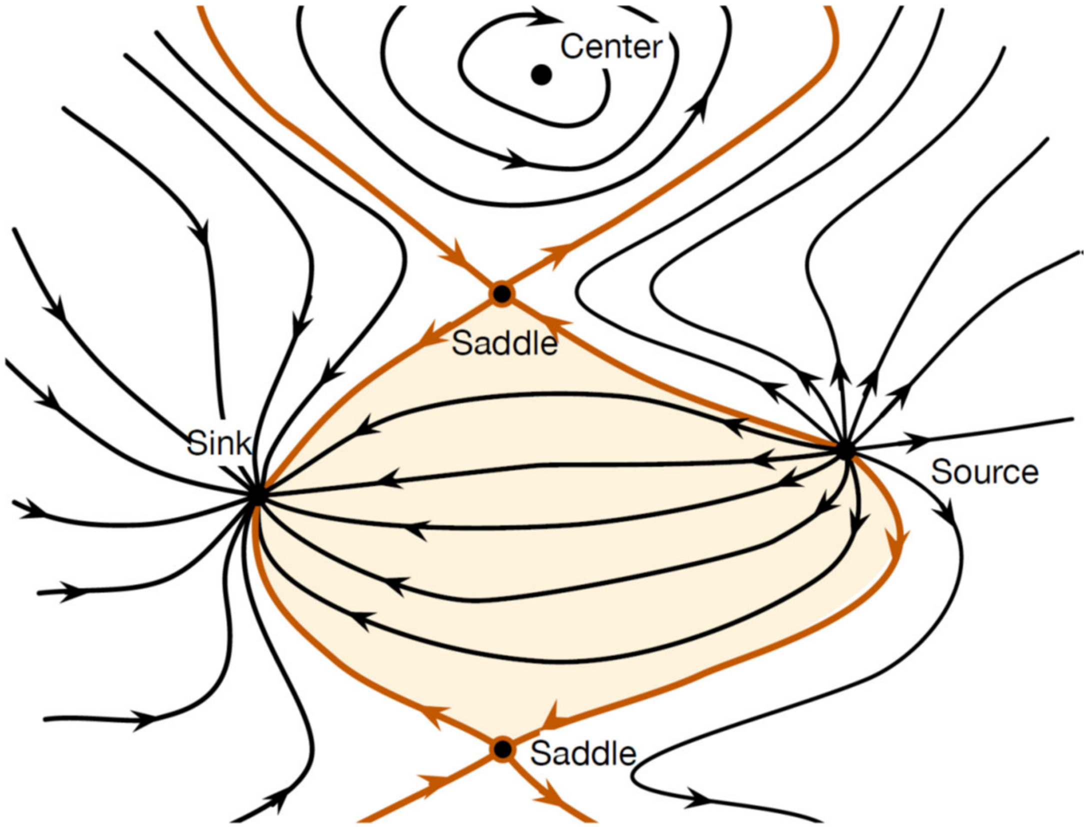




Sources (red), sinks (blue), saddles (yellow)

Vector Field Topology: Topological Skeleton

Connect critical points by *separatrices*



Index of Critical Points / Vector Fields

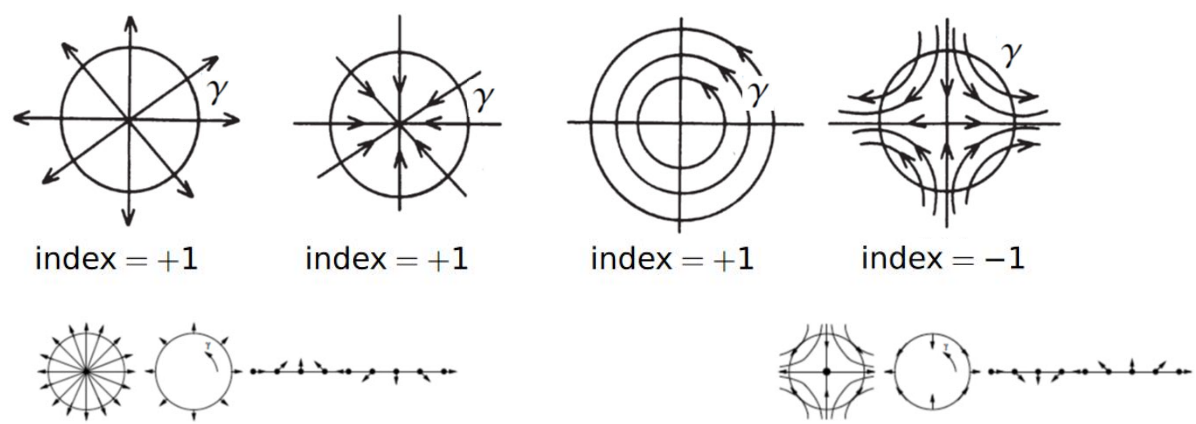
Poincaré index (in scalar field topology we have the *Morse index*)

- Can compute index (winding number) for each critical point
- Index of a region is the sum of the critical point indexes inside
- Sum of all indexes over a manifold is its Euler characteristic

Do a loop (Jordan curve) around each critical point: the index is its (Brouwer) degree: integer how often the vector field along the loop turns around (determined by angle 1-form integrated over oriented 1-manifold)

$$\text{index}_\gamma = \frac{1}{2\pi} \oint_\gamma d\alpha$$

$$\alpha = \arctan \frac{v}{u}$$

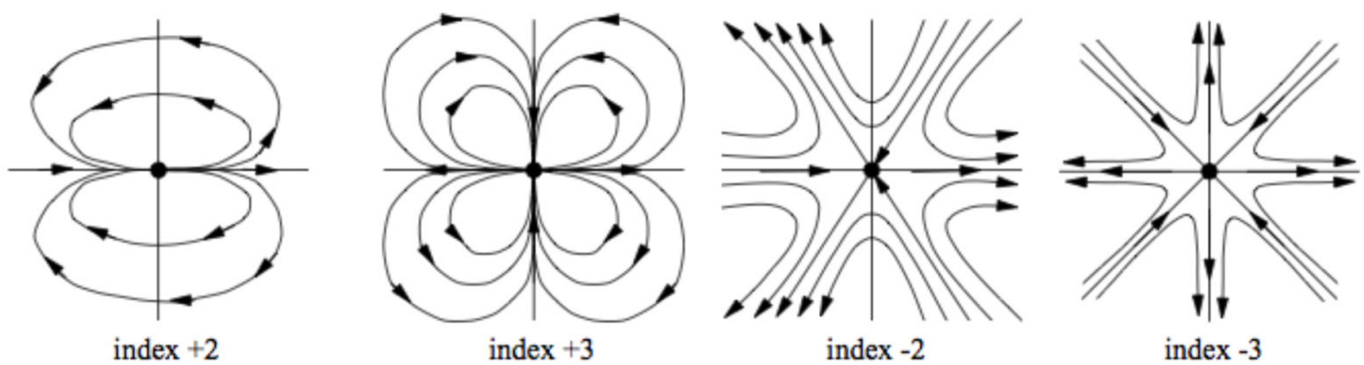


Higher-Order Critical Points

Higher than first-order

- Sectors can by elliptic, parabolic, hyperbolic
- For index sum over number of elliptic and hyperbolic sectors

$$\text{index}_{cp} = 1 + \frac{n_e - n_h}{2}$$



(dipole)

(see monkey saddle)

Example: Differential Topology

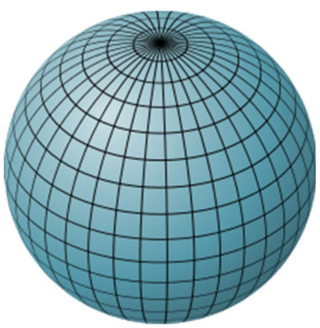
Topological information from vector fields on manifold

- Independent of actual vector field! Poincaré-Hopf theorem (sum of indexes == Euler char.)
- Useful constraints: vector field editing, simplification, sphere always has critical point, …

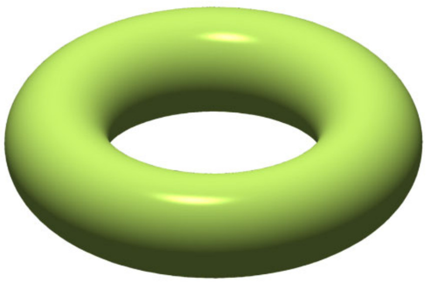
Topological invariant: Euler characteristic $\chi(M)$ of manifold M

(for 2-manifold mesh: $\chi(M) = V - E + F$)

$\chi = 2 - 2g$ (orientable)



genus $g = 0$
Euler characteristic $\chi = 2$

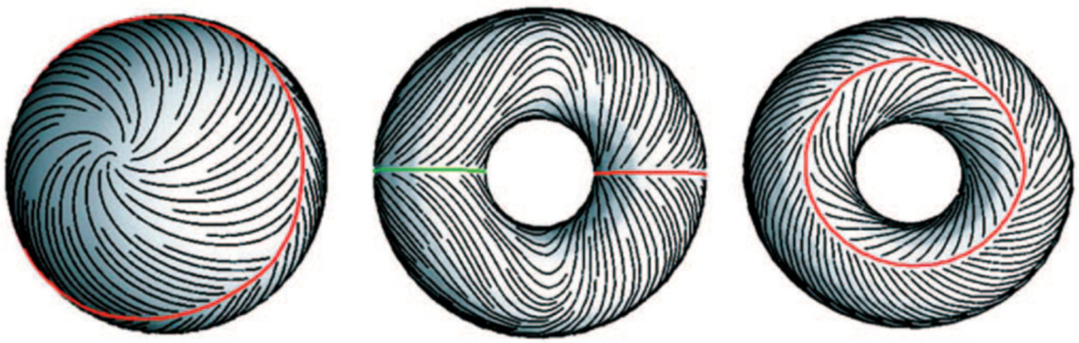


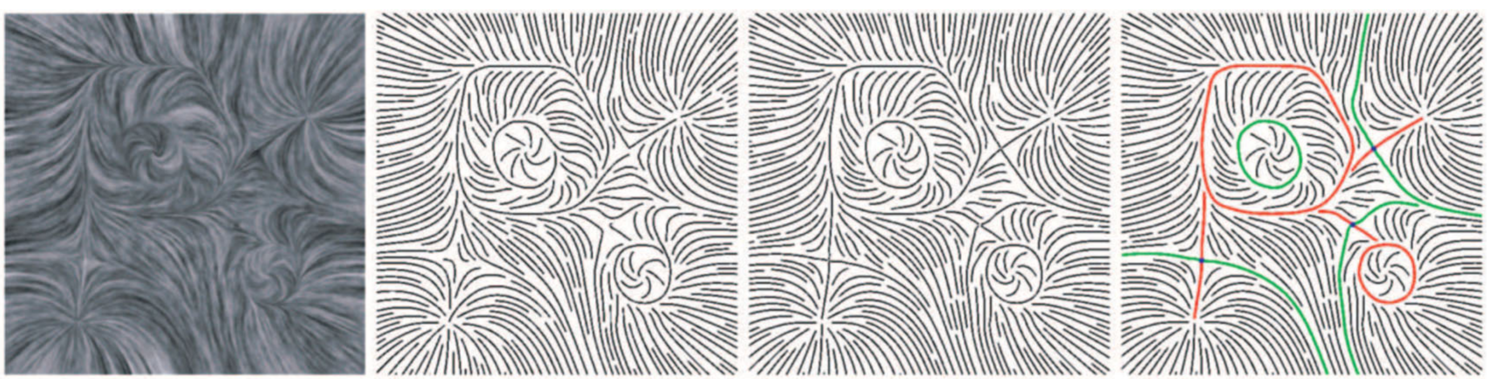
genus $g = 1$
Euler characteristic $\chi = 0$

genus $g = 2$
Euler characteristic $\chi = -2$

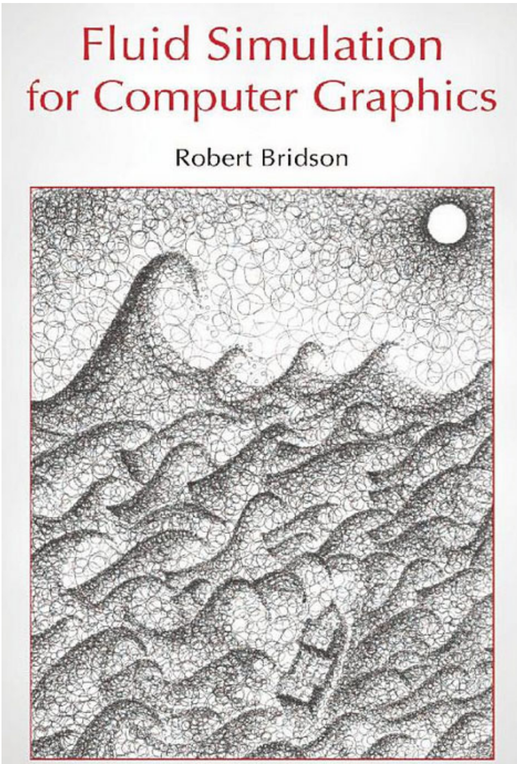
Example: Vector Field Editing

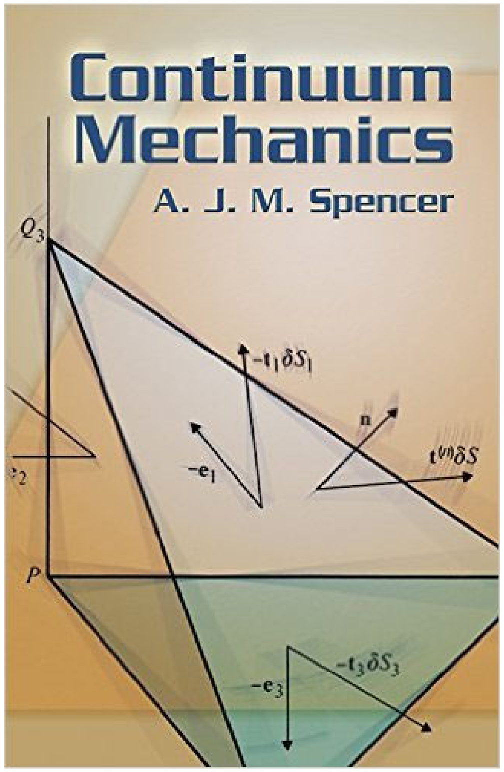
Guoning Chen et al., Vector Field Editing and Periodic Orbit Extraction Using Morse Decomposition, IEEE TVCG, 2007

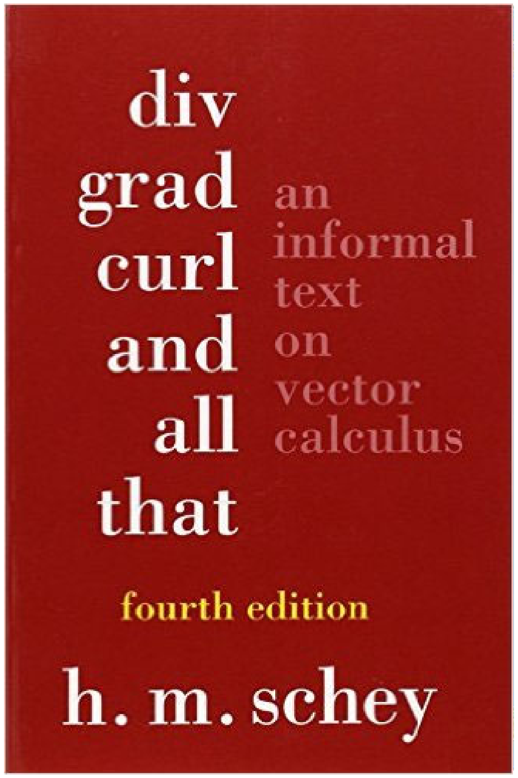




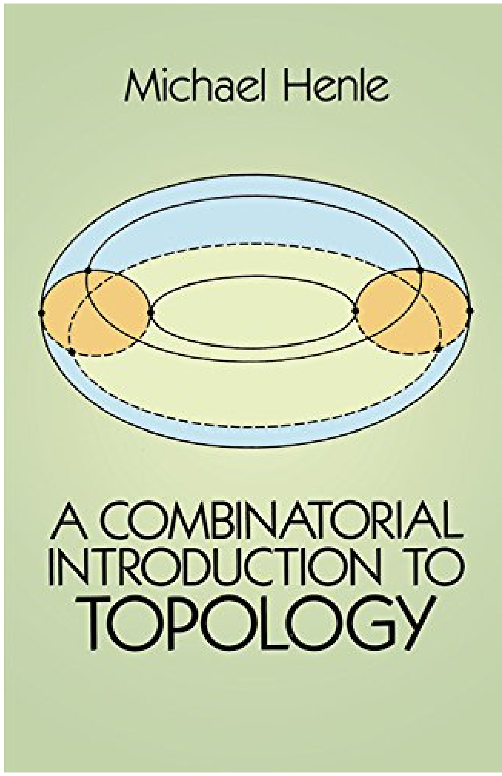
Recommended Books (1)

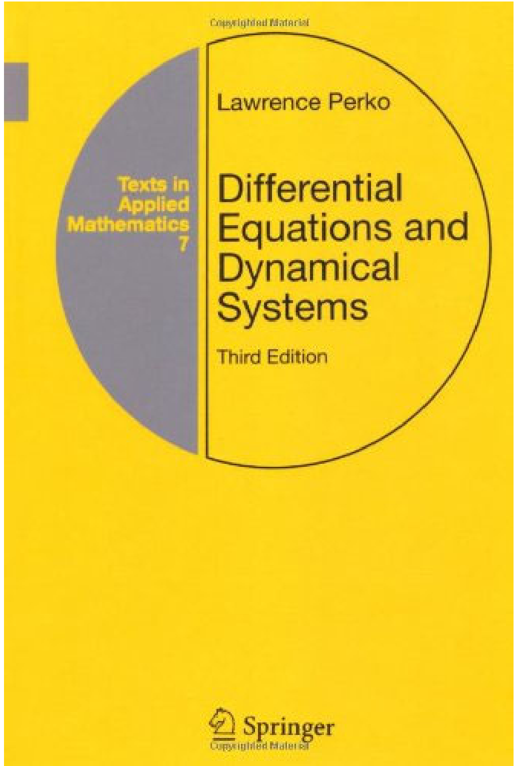


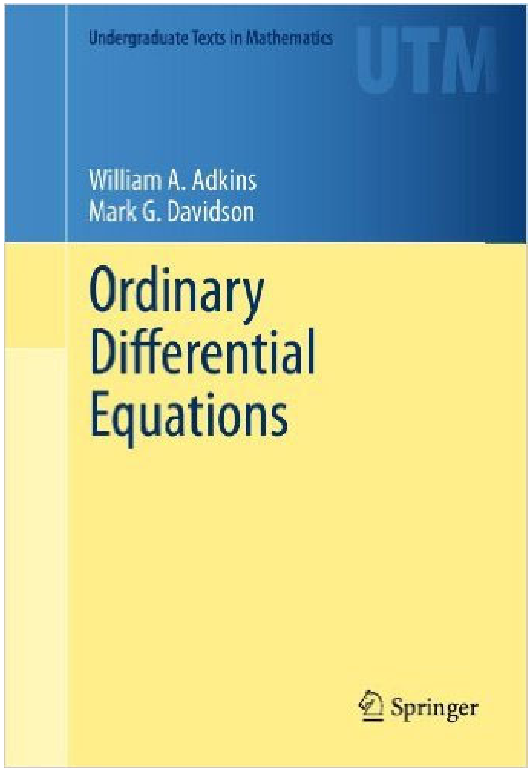




Recommended Books (2)







Bonus: Classification of Critical Points in Scalar Fields

Critical Points of Scalar Fields

For a scalar field $f : M \to \mathbb{R}$ the *critical points* are where the differential df vanishes (also meaning ∇f vanishes)

$$\boxed{df = 0} \qquad df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \quad = 0$$

Critical point is *non-degenerate* if the Hessian does not vanish

For non-degenerate critical points

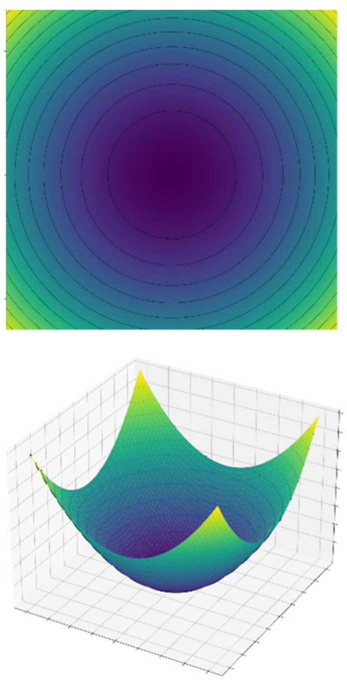
- Critical point is isolated
- Hessian matrix determines Morse index of critical point

General Case (2D Scalar Fields)

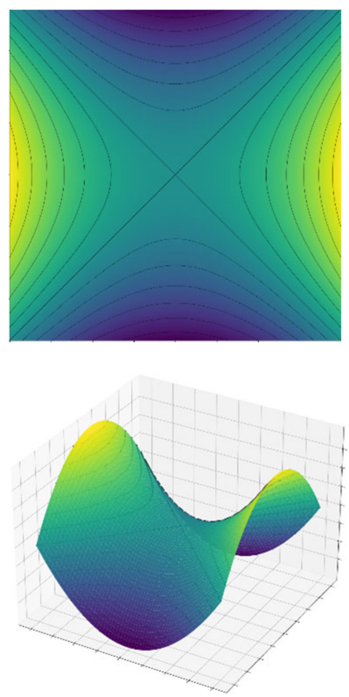
In 2D scalar fields, only *three types* of (isolated, non-degenerate) critical points

Index of critical point: dimension of eigenspace with negative-definite Hessian

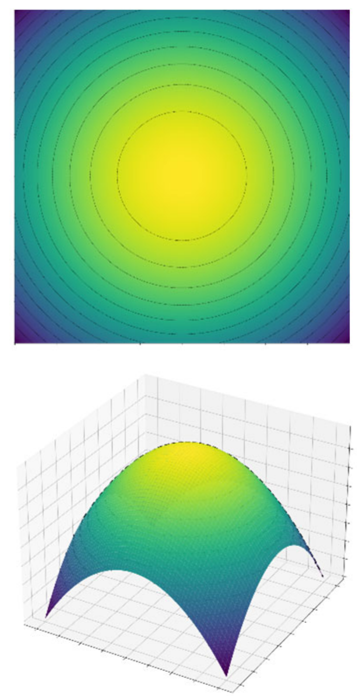
minimum
(index 0)



saddle point
(index 1)



maximum
(index 2)



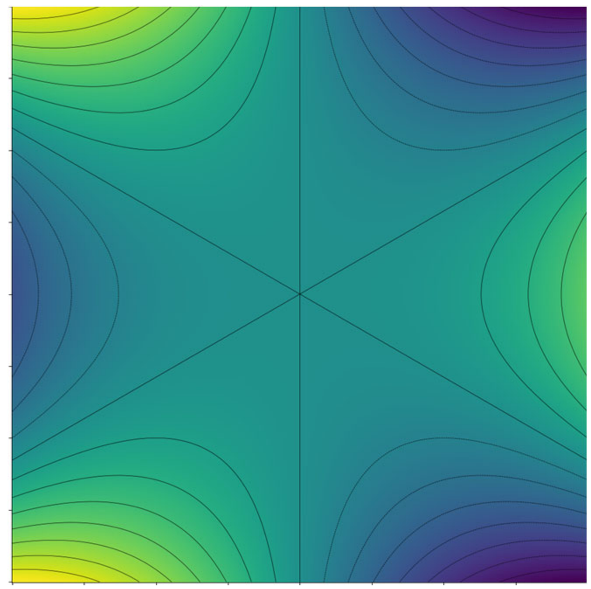
Interesting Degenerate Critical Points?

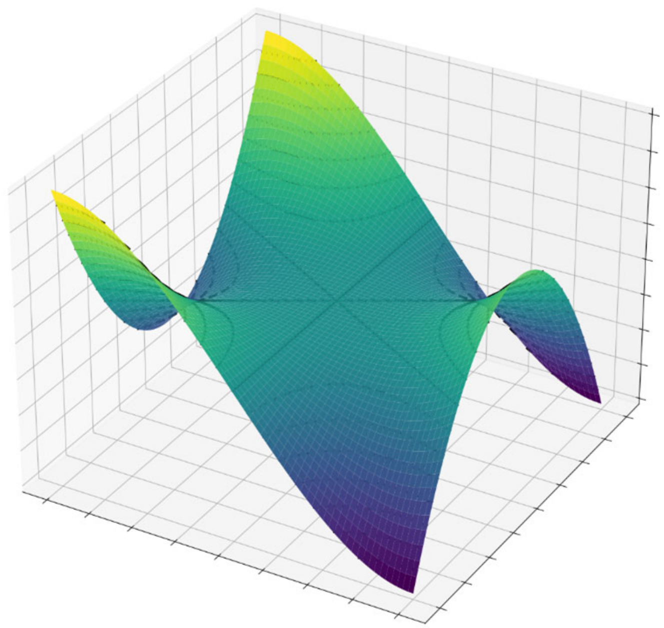
Hessian matrix is singular (determinant = 0)

- Cannot say what happens: need higher-order derivatives, …

Interesting example: monkey saddle $z = x^3 - 3xy^2$ ('third-order saddle')

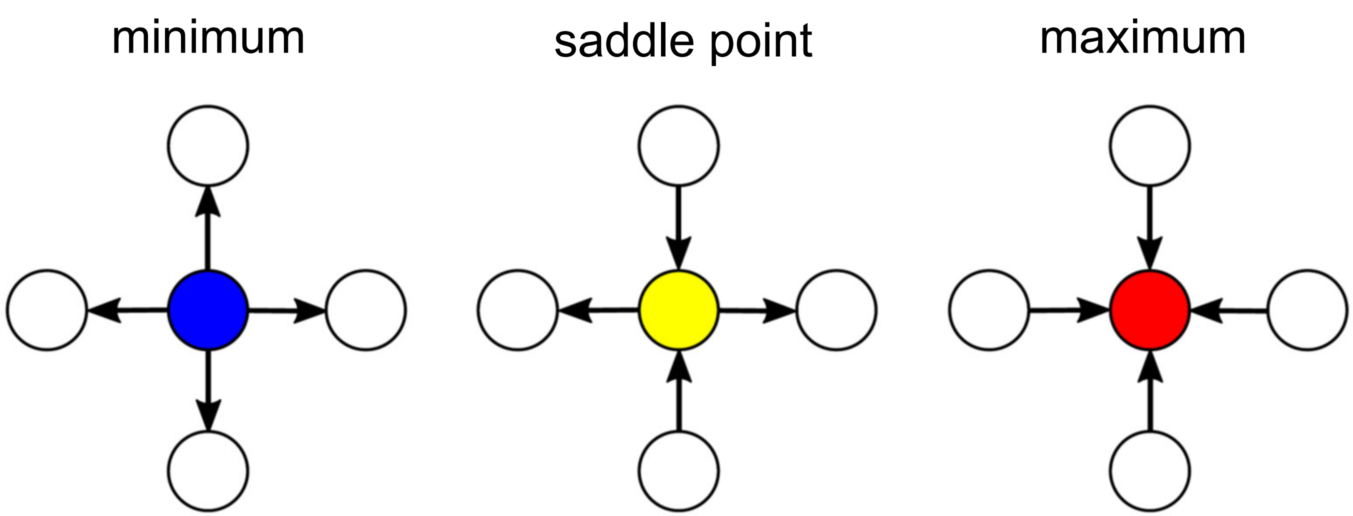
- Point (0,0) in center: Hessian = 0; Gaussian curvature = 0 (umbilical point)





Discrete Classification of Critical Points

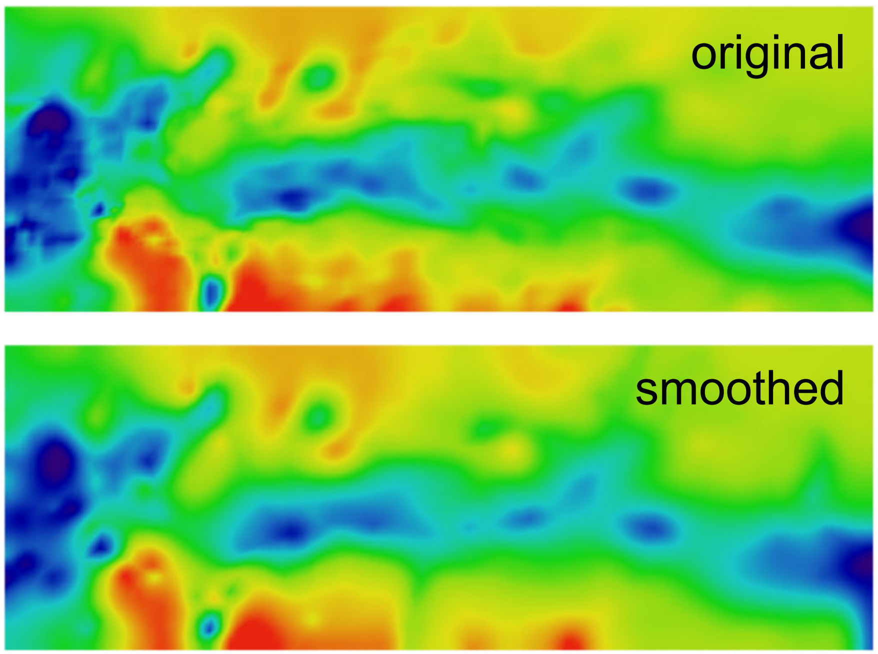
Combinatorial classification (looking at and comparing neighbors)
instead of looking at derivatives
(i.e., derivatives of the smooth function that is not known)

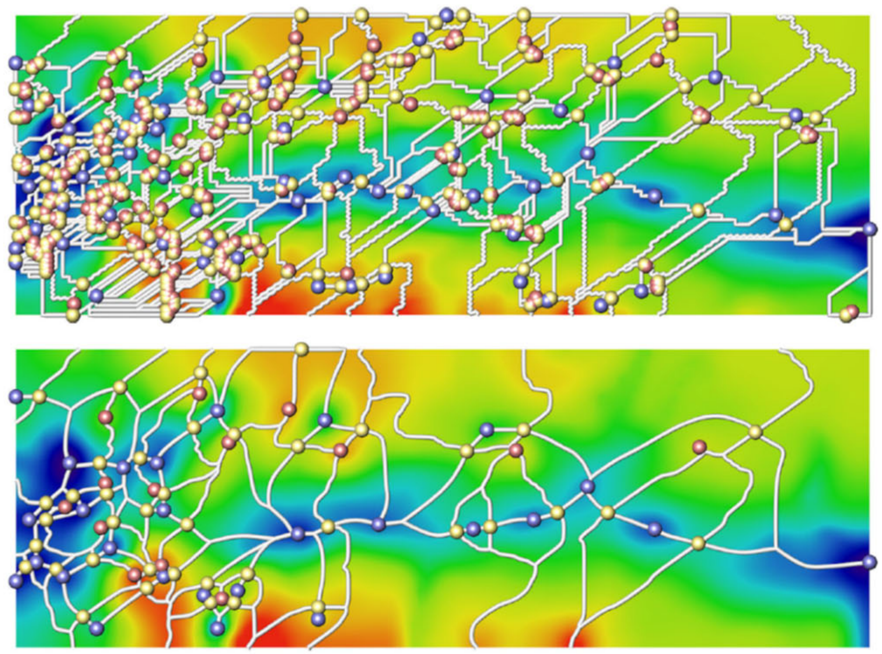


…toward scalar field topology, discrete Morse theory, Morse-Smale complex, …

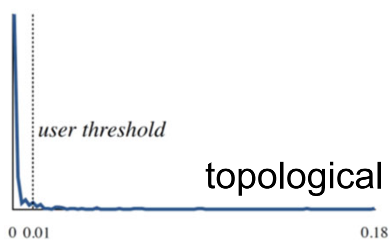
Example: Scalar Field Simplification

Topology-based smoothing of 2D scalar fields, Weinkauf et al., 2010





Morse-Smale complex



topological persistence

Example: Differential Topology

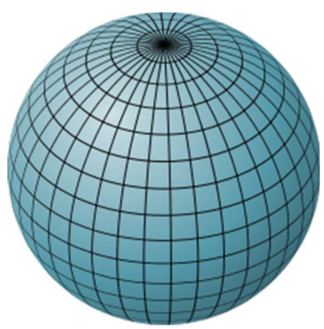
Morse theory

- Morse function: scalar function where all critical points are non-degenerate and have different critical value

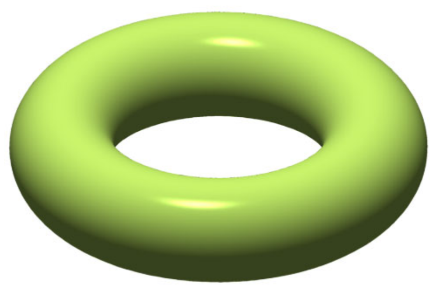
Topological invariant: Euler characteristic $\chi(M)$ of manifold M

(for 2-manifold mesh: $\chi(M) = V - E + F$)

$\chi = 2 - 2g$ (orientable)



genus $g = 0$
Euler characteristic $\chi = 2$



genus $g = 1$
Euler characteristic $\chi = 0$

genus $g = 2$
Euler characteristic $\chi = -2$

Example: Differential Topology

Morse theory

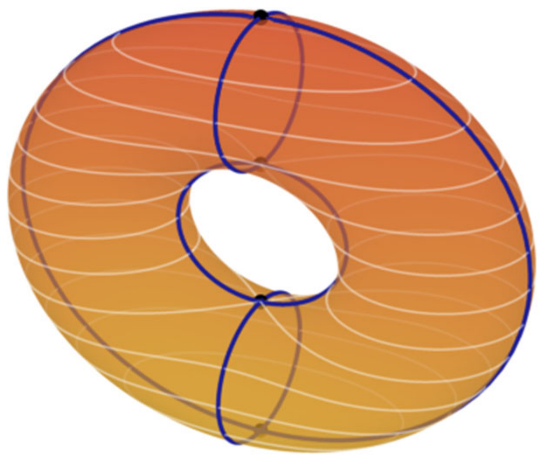
- Morse function: scalar function where all critical points are non-degenerate and have different critical value

Topological invariant: Euler characteristic $\chi(M)$ of manifold M

$$\chi(M) = \sum_{i=0}^{n} (-1)^i m_i$$

m_i: number of critical points with index i

n: dimensionality of M



genus $g(M) = 1$

Euler characteristic $\chi(M) = 0 \; (= 1 - 2 + 1)$

scalar function on torus is height function $f(x,y,z) = z$:

1 min, 1 max, 2 saddles

critical points are where

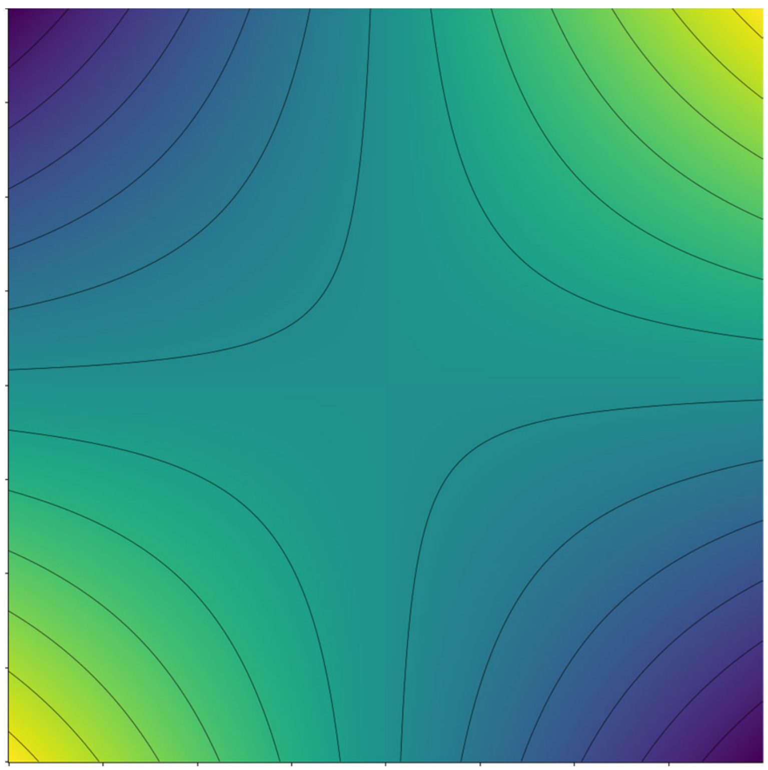
$$df(x,y,z) = 0$$

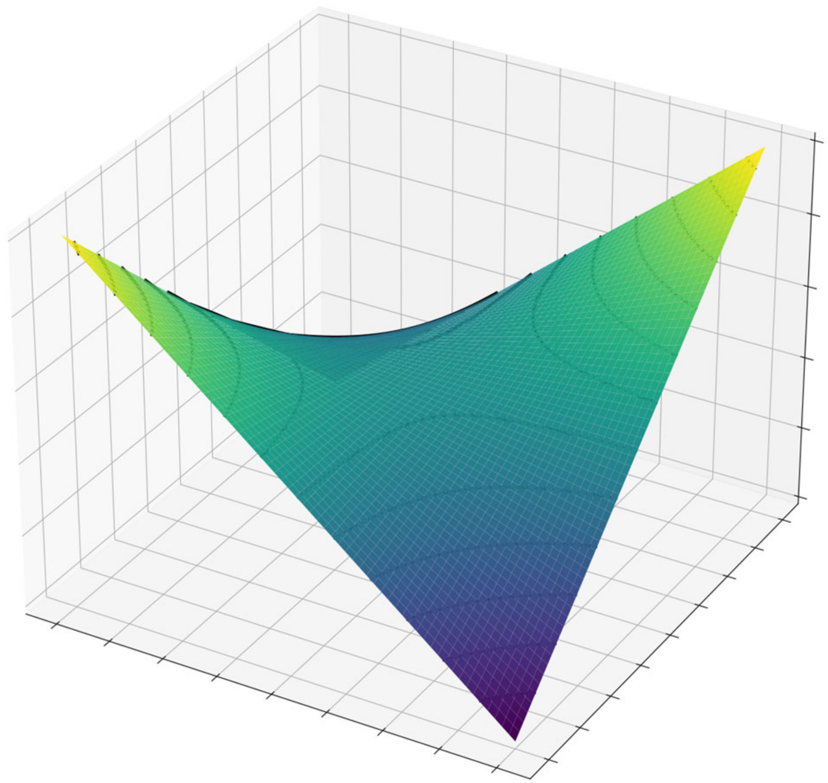
(tangent plane horizontal)

Remember This One? Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers)

Example #1: 1 at bottom-left and top-right, 0 at top-left and bottom-right

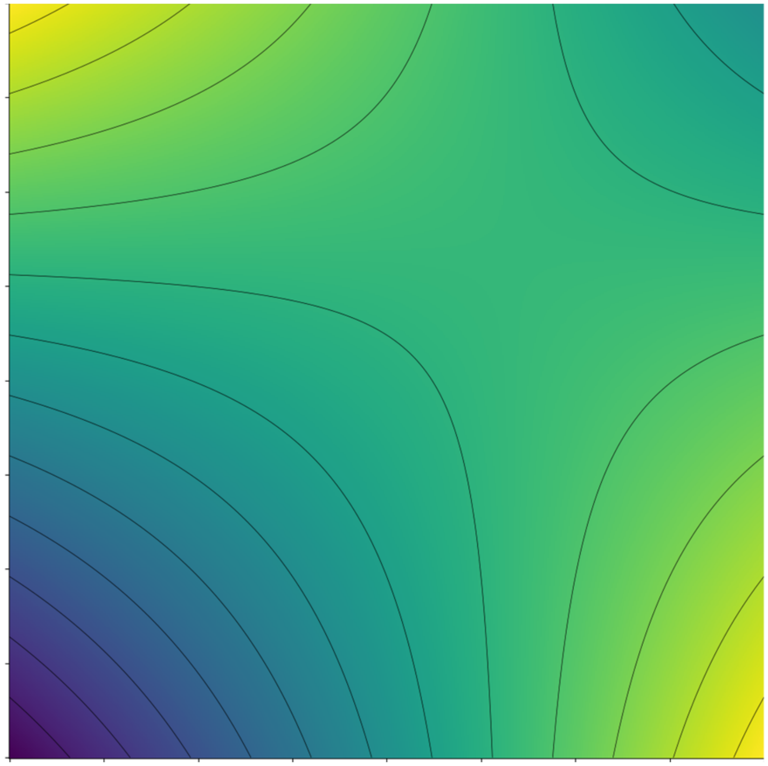


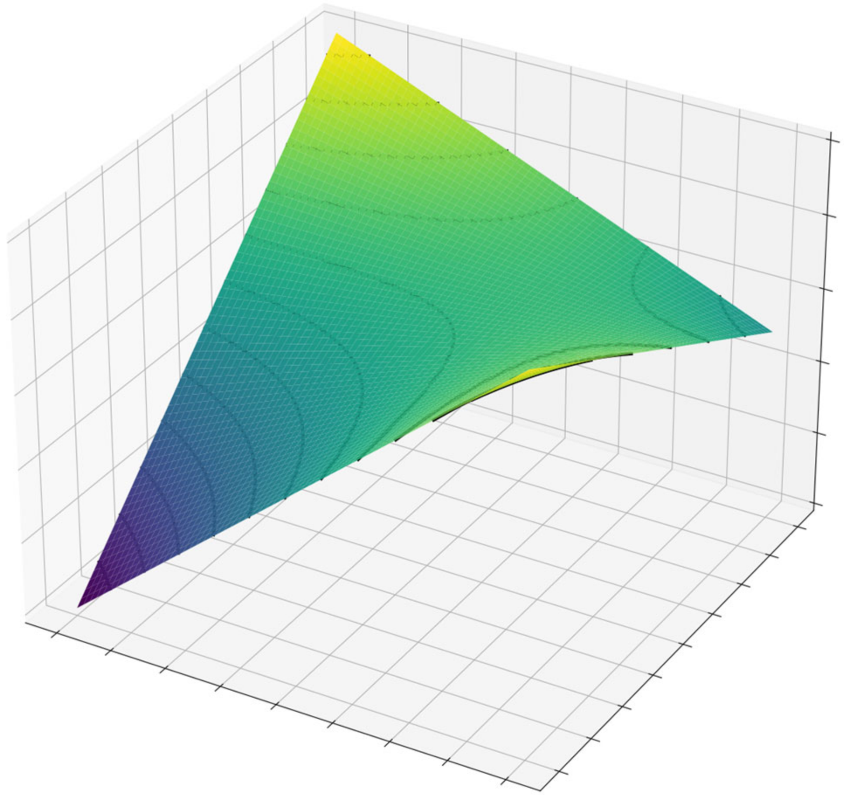


Remember This One? Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers)

Example #2: 1 at top-left and bottom-right, 0 at bottom-left, 0.5 at top-right





Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

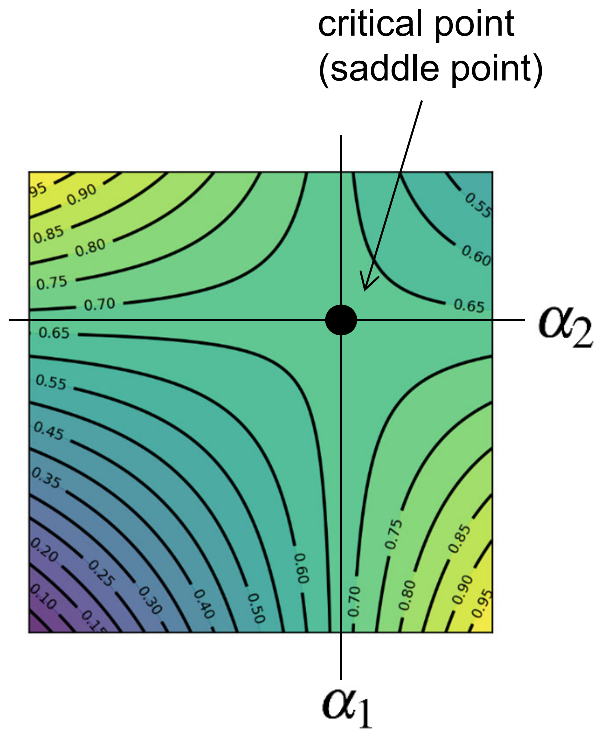
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0: \qquad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$



Bi-Linear Interpolation: Critical Points

Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

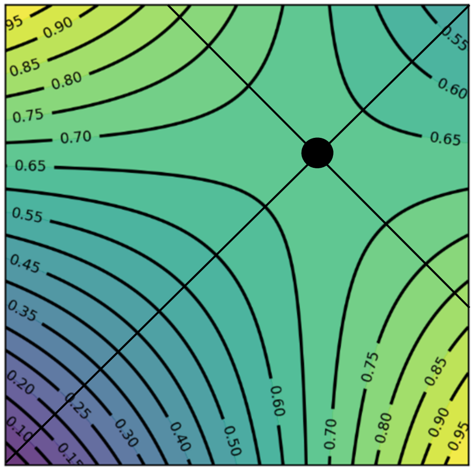
$$\begin{bmatrix} \frac{\partial^2 f}{\partial \alpha_1^2} & \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

$$a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



(here also: principal curvature magnitudes and directions
of this function’s graph == surface embedded in 3D)

Bi-Linear Interpolation: Critical Points

Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

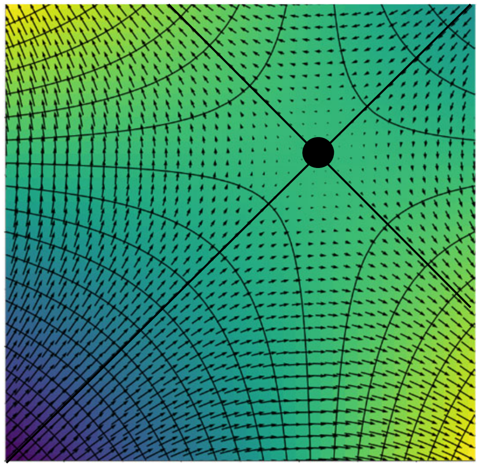
$$\begin{bmatrix} \frac{\partial^2 f}{\partial \alpha_1^2} & \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

$$a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



(here also: principal curvature magnitudes and directions
of this function's graph == surface embedded in 3D)

Interlude: Curvature and Shape Operator

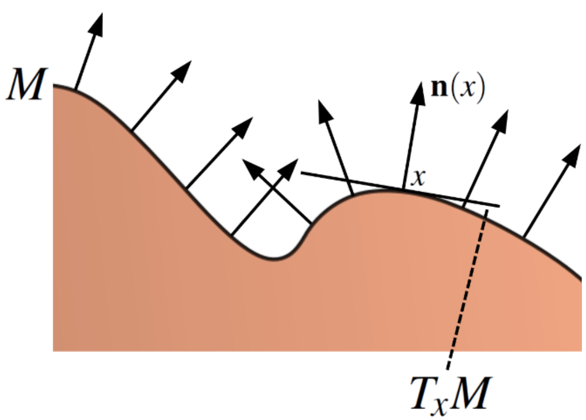
Gauss map

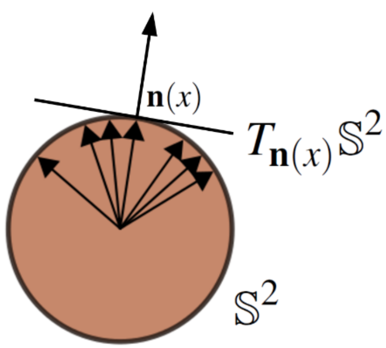
$$\mathbf{n}: M \to \mathbb{S}^2$$
$$x \mapsto \mathbf{n}(x)$$

Differential of Gauss map

$$d\mathbf{n}: TM \to T\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

$$(d\mathbf{n})_x: T_xM \to T_{\mathbf{n}(x)}\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$





Shape operator (Weingarten map)

$$\mathbf{S}: TM \to TM$$

$$\mathbf{S}_x: T_xM \to T_xM$$
$$\mathbf{v} \mapsto \mathbf{S}_x(\mathbf{v}) = d\mathbf{n}(\mathbf{v})$$

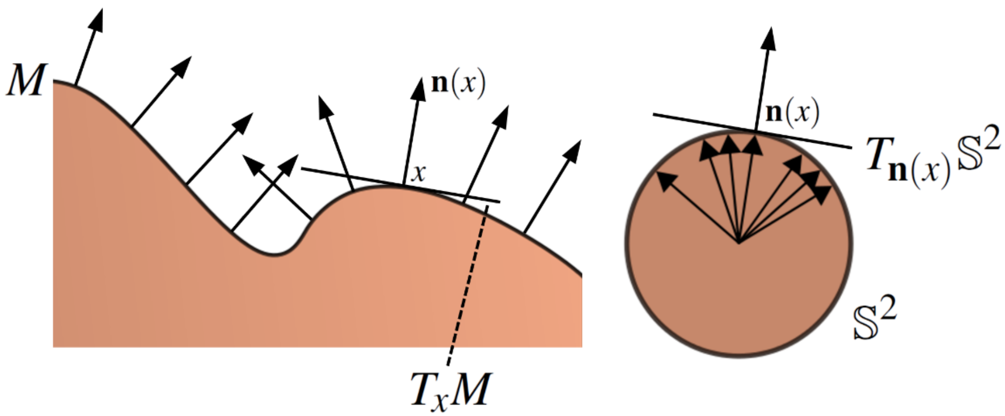
Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**; Determinant is Gaussian curvature

$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_xM$$

Interlude: Curvature and Shape Operator

Gauss map

$$\mathbf{n}: M \to \mathbb{S}^2$$
$$x \mapsto \mathbf{n}(x)$$



Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**; Determinant is Gaussian curvature

$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_x M$$

Differential of Gauss map

$$d\mathbf{n}: TM \to T\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

$$(d\mathbf{n})_x: T_x M \to T_{\mathbf{n}(x)}\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

Shape operator (Weingarten map)

$$\mathbf{S}: TM \to TM$$

$$\mathbf{S}_x: T_x M \to T_x M$$
$$\mathbf{v} \mapsto \mathbf{S}_x(\mathbf{v}) = \nabla_{\mathbf{v}}\mathbf{n}$$

Interlude: Curvature and Shape Operator

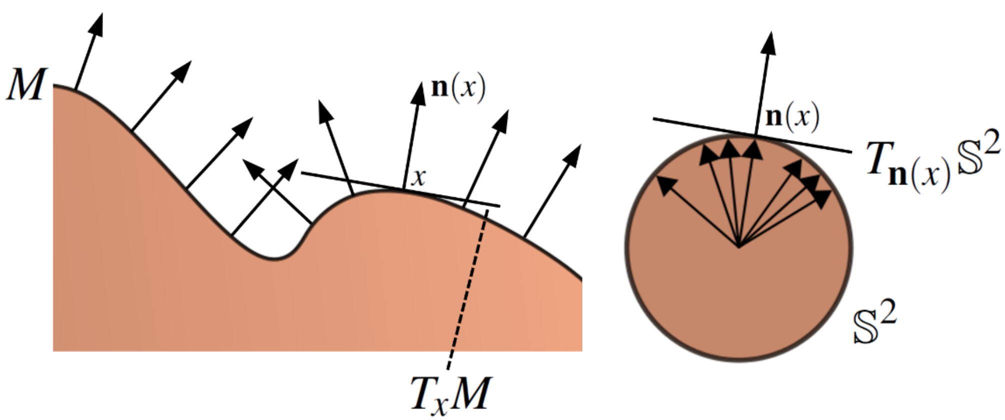
Gauss map

$$\mathbf{n} : M \to \mathbb{S}^2$$
$$x \mapsto \mathbf{n}(x)$$

Differential of Gauss map

$$d\mathbf{n} : TM \to T\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

$$(d\mathbf{n})_x : T_xM \to T_{\mathbf{n}(x)}\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$



Shape operator (Weingarten map)

$$\mathbf{S} : TM \to TM$$

$$\mathbf{S}_x : T_xM \to T_xM$$
$$\mathbf{v} \mapsto \mathbf{S}_x(\mathbf{v}) = -\nabla_{\mathbf{v}}\mathbf{n}$$

(sign is convention)

Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**; Determinant is Gaussian curvature

$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_xM$$

Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama