CS 247 – Scientific Visualization
Lecture 26: Vector / Flow Visualization, Pt. 5

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Reading Assignment #13 (until May 7)

Read (required):

- Data Visualization book
  - Chapter 6.1 (Divergence and Vorticity)
  - Chapter 6.6 (Texture-Based Vector Visualization)
- Diffeomorphisms / smooth deformations
  `https://en.wikipedia.org/wiki/Diffeomorphism`
- Learn how convolution (the convolution of two functions) works:
  `https://en.wikipedia.org/wiki/Convolution`
- B. Cabral, C. Leedom:
  Imaging Vector Fields Using Line Integral Convolution, SIGGRAPH 1993
  `http://dx.doi.org/10.1145/166117.166151`

Quiz #3: May 7

Organization

- First 30 min of lecture
- No material (book, notes, ...) allowed

Content of questions

- Lectures (both actual lectures and slides)
- Reading assignments (except optional ones)
- Programming assignments (algorithms, methods)
- Solve short practical examples

Vector Fields, Vector Calculus, and Dynamical Systems

Fluid Simulation: Navier Stokes Equations

Incompressible (divergence-free) Navier Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F},$$

$$\nabla \cdot \mathbf{u} = 0,$$

Components:

- Self-advection of velocity (i.e., advection of velocity according to velocity)
- Pressure gradient (force due to pressure differences)
- Diffusion of velocity due to viscosity (for viscous fluids, i.e., not inviscid)
- Application of (arbitrary) external forces, e.g., gravity, user input, etc.

Some Vector Calculus (1)

Gradient (scalar field → vector field)

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \right)$$

- Direction of steepest ascent; magnitude = rate
- *Conservative* vector field: gradient of some scalar (potential) function

Divergence (vector field → scalar field)

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

- Volume density of outward flux:
  "exit rate: source? sink?"
- *Incompressible/solenoidal/divergence-free vector field*: div u = 0
  can express as curl (next slide) of some vector (potential) function

Laplacian (scalar field → scalar field)

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}$$

- Divergence of gradient
- Measure for difference between point and its neighborhood

Some Vector Calculus (2)

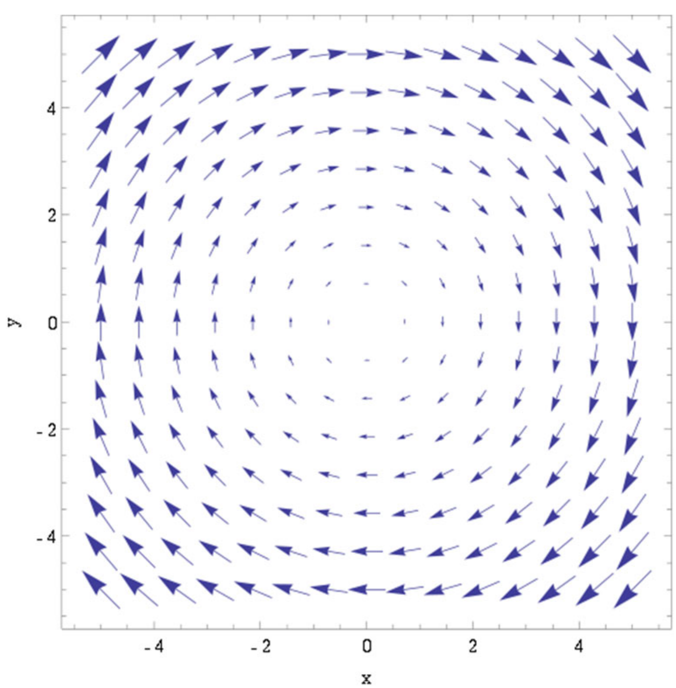
Curl (vector field → vector field)

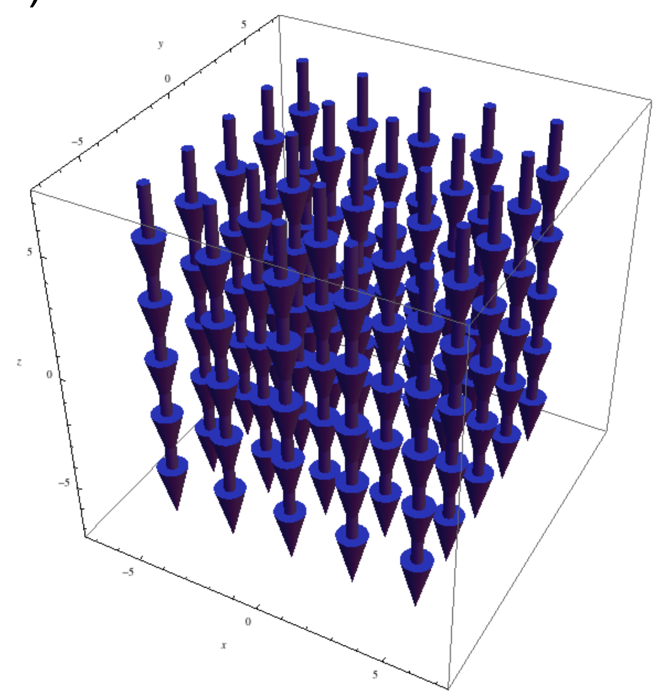
- Circulation density at a point (*vorticity*)
- If curl vanishes everywhere: irrotational/curl-free field
- Every conservative (path-independent) field is irrotational (and vice versa if domain is simply connected)

$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

these are partial derivatives!

Example:
curl = const everywhere





Some Vector Calculus (3)

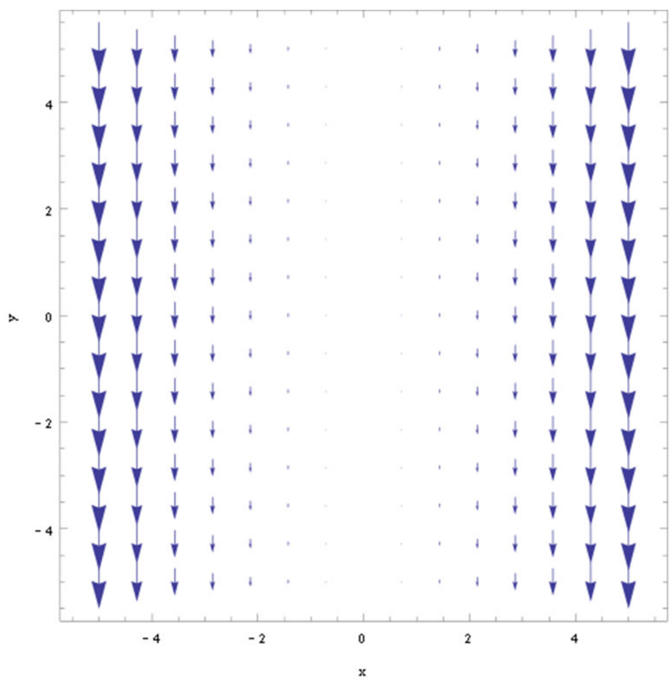
Curl (vector field → vector field)

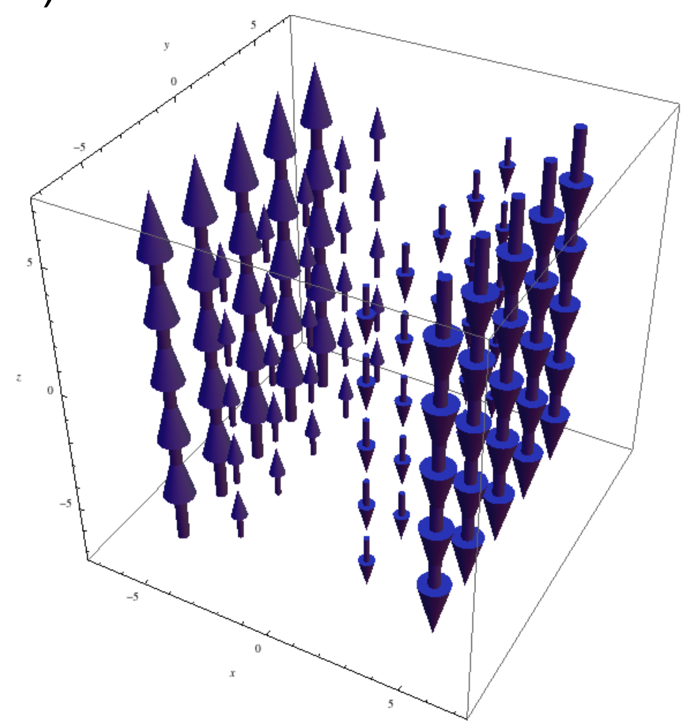
- Circulation density at a point (*vorticity*)
- If curl vanishes everywhere: irrotational/curl-free field
- Every conservative (path-independent) field is irrotational (and vice versa if domain is simply connected)

$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

these are partial derivatives!

Example: curl not always "obviously rotational"





Some Vector Calculus (4)

Curl (vector field → vector field)

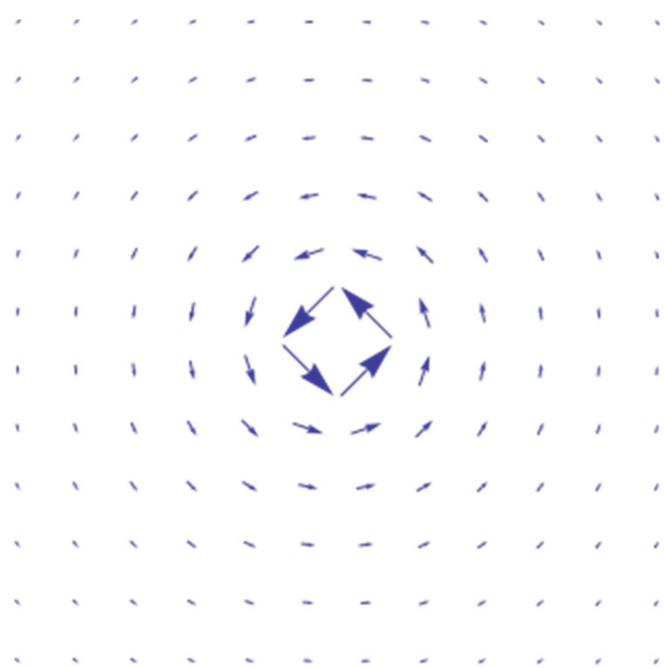
- Circulation density at a point (*vorticity*)
- If curl vanishes everywhere: irrotational/curl-free field
- Every conservative (path-independent) field is irrotational (and vice versa if domain is simply connected)

$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

these are partial derivatives!

Example:
non-obvious
curl-free field

[this domain is **not** simply connected! it is the “punctured plane”, i.e., the point (0,0) is not in the domain]



$$\mathbf{v}(x,y,z) = \frac{(-y,x,0)}{x^2+y^2}$$

not defined at (x,y) = (0,0)

$$v_x = u_y \qquad \nabla \times \mathbf{v} = 0$$

velocity gradient $\nabla \mathbf{v}$ is symmetric (see later)

Some Vector Calculus (5)

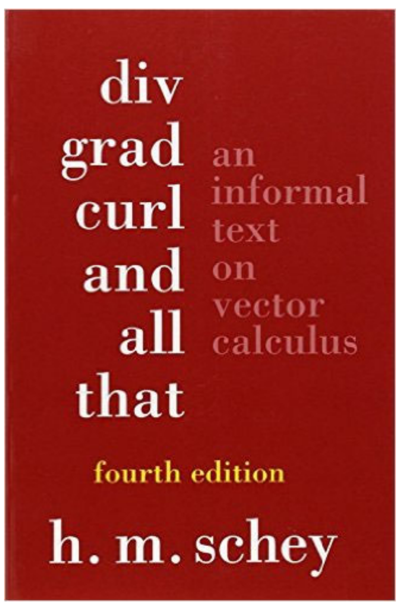
Curl (vector field → vector field)

- Circulation density at a point (*vorticity*)
- If curl vanishes everywhere: irrotational/curl-free field
- Every conservative (path-independent) field is irrotational (and vice versa if domain is simply connected)

$$\nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

these are partial derivatives!

Book:



Interactive tutorial on curl:
`http://mathinsight.org/curl_idea`

Fundamental theorem of vector calculus:
Helmholtz decomposition: Any vector field can be expressed as the sum of a solenoidal (*divergence-free*) vector field and an irrotational (*curl-free*) vector field (Helmholtz-Hodge: plus *harmonic* vector field)

Vector Fields and Dynamical Systems (1)

Velocity gradient tensor, (vector field → tensor field)

- Gradient of vector field: how does the vector field change?
- In Cartesian coordinates: *spatial partial derivatives (Jacobian matrix)*

$$\nabla \mathbf{v}\,(x,y,z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

these are partial derivatives!

- Can be decomposed into *symmetric* part + *anti-symmetric* part

$$\nabla \mathbf{v} = \mathbf{D} + \mathbf{S}$$ *velocity gradient tensor*

sym.: $\mathbf{D} = ½\,(\,\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}\,)$ deform.: *rate-of-strain tensor*

skew-sym.: $\mathbf{S} = ½\,(\,\nabla \mathbf{v} - (\nabla \mathbf{v})^{\mathrm{T}}\,)$ rotation: *vorticity/spin tensor*

Vector Fields and Dynamical Systems (2)

Vorticity/spin/angular velocity tensor

- Antisymmetric part of velocity gradient tensor
- Corresponds to vorticity/curl/angular velocity (beware of factor ½)

$$\mathbf{S} = ½ \, (\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathrm{T}})$$

these are partial derivatives!

$$\mathbf{S} = ½ \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

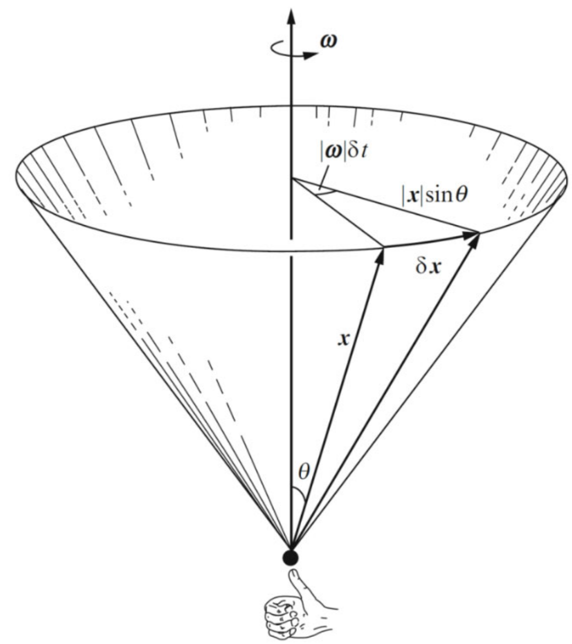
$\mathbf{S}$ acts on vector like cross product with ω : $\mathbf{S} \bullet \; = ½ \, \omega \times$

$$\mathbf{v}^{(r)} = \mathbf{S} \cdot d\mathbf{r} = \frac{1}{2} \left[\nabla \mathbf{v} - (\nabla \mathbf{v})^T \right] \cdot d\mathbf{r} = \frac{1}{2} \boldsymbol{\omega} \times d\mathbf{r}$$

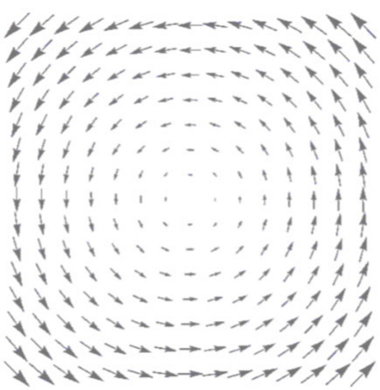
Angular Velocity of Rigid Body Rotation

Rate of rotation

- Scalar ω: angular displacement per unit time (rad s^{-1})
  - Angle Θ at time t is $\Theta(t) = \omega t$; $\omega = 2\pi f$ where f is the frequency ($f = 1/T$; s^{-1})
- Vector $\boldsymbol{\omega}$: axis of rotation; magnitude is angular speed (if $\boldsymbol{\omega}$ is curl: speed x2)
  - Beware of different conventions that differ by a factor of ½ !



Cross product of ½$\boldsymbol{\omega}$ with vector to center of rotation ($\mathbf{r}$) gives linear velocity vector $\mathbf{v}$ (tangent)



$$\mathbf{v}^{(r)} = \frac{1}{2}\boldsymbol{\omega} \times d\mathbf{r}$$

Velocity Gradient Tensor and Components (1)

Velocity gradient tensor

(here: in Cartesian coordinates)

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial}{\partial x} v^x & \frac{\partial}{\partial y} v^x & \frac{\partial}{\partial z} v^x \\ \frac{\partial}{\partial x} v^y & \frac{\partial}{\partial y} v^y & \frac{\partial}{\partial z} v^y \\ \frac{\partial}{\partial x} v^z & \frac{\partial}{\partial y} v^z & \frac{\partial}{\partial z} v^z \end{bmatrix}$$

these are the same partial derivatives as before!

$$\nabla \mathbf{v} = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right) + \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^T\right)$$

Velocity Gradient Tensor and Components (2)

Rate-of-strain (rate-of-deformation) tensor

(symmetric part; here: in Cartesian coordinates)

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} 2\frac{\partial}{\partial x}v^x & \frac{\partial}{\partial y}v^x + \frac{\partial}{\partial x}v^y & \frac{\partial}{\partial z}v^x + \frac{\partial}{\partial x}v^z \\ \frac{\partial}{\partial x}v^y + \frac{\partial}{\partial y}v^x & 2\frac{\partial}{\partial y}v^y & \frac{\partial}{\partial z}v^y + \frac{\partial}{\partial y}v^z \\ \frac{\partial}{\partial x}v^z + \frac{\partial}{\partial z}v^x & \frac{\partial}{\partial y}v^z + \frac{\partial}{\partial z}v^y & 2\frac{\partial}{\partial z}v^z \end{bmatrix}$$

$$tr(\mathbf{D}) = \nabla \cdot \mathbf{v}$$

Velocity Gradient Tensor and Components (3)

Vorticity tensor (spin tensor)

(skew-symmetric part; here: in Cartesian coordinates)

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial}{\partial y}v^x - \frac{\partial}{\partial x}v^y & \frac{\partial}{\partial z}v^x - \frac{\partial}{\partial x}v^z \\ \frac{\partial}{\partial x}v^y - \frac{\partial}{\partial y}v^x & 0 & \frac{\partial}{\partial z}v^y - \frac{\partial}{\partial y}v^z \\ \frac{\partial}{\partial x}v^z - \frac{\partial}{\partial z}v^x & \frac{\partial}{\partial y}v^z - \frac{\partial}{\partial z}v^y & 0 \end{bmatrix}$$

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \qquad \boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$$

Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama