CS 247 – Scientific Visualization
Lecture 24: Vector / Flow Visualization, Pt. 3

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Reading Assignment #12 (until Apr 30)

Read (required):

- Data Visualization book
  - Chapter 6 (Vector Visualization)
    - Beginning (before 6.1)
    - Chapters 6.2, 6.3, 6.5
- More general vector field basics (the book is not very precise on the basics)

  `https://en.wikipedia.org/wiki/Vector_field`

Read (optional):

- Paper:
  Bruno Jobard and Wilfrid Lefer
  Creating Evenly-Spaced Streamlines of Arbitrary Density,

  `http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.29.9498`

Vector fields as ODEs

For simplicity, the vector field is now interpreted as a velocity field.

Then the field $\mathbf{v}(\mathbf{x},t)$ describes the connection between location and velocity of a (massless) particle.

It can equivalently be expressed as an ordinary differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t),t)$$

This ODE, together with an initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0 ,$$

is a so-called initial value problem (IVP).

Its solution is the integral curve (or trajectory)

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(\tau),\tau)\,d\tau$$

The integral curve is a pathline, describing the path of a massless particle which was released at time t_0 at position x_0.

Remark: $t < t_0$ is allowed.

For static fields, the ODE is autonomous:

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t))$$

and its integral curves

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(\tau))\,d\tau$$

are called field lines, or (in the case of velocity fields) streamlines.

In static vector fields, pathlines and streamlines are identical.

In time-dependent vector fields, instantaneous streamlines can be computed from a "snapshot" at a fixed time T (which is a static vector field)

$$\mathbf{v}_T(\mathbf{x}) = \mathbf{v}(\mathbf{x}, T)$$

In practice, time-dependent fields are often given as a dataset per time step. Each dataset is then a snapshot.

Streamline integration

Outline of algorithm for numerical streamline integration (with obvious extension to pathlines):

Inputs:

- static vector field $\mathbf{v}(\mathbf{x})$
- seed points with time of release $(\mathbf{x}_0, t_0)$
- control parameters:
  - step size (temporal, spatial, or in local coordinates)
  - step count limit, time limit, etc.
  - order of integration scheme

Output:

- streamlines as "polylines", with possible attributes (interpolated field values, time, speed, arc length, etc.)

Preprocessing:

- set up search structure for point location
- for each seed point:
  - global point location: Given a point **x**,
    find the cell containing **x** and the local coordinates (ξ, η, ζ)
    or ir the grid is structured:
    find the computational space coordinates $(i + \xi,\ j + \eta,\ k + \zeta)$
  - If **x** is not found in a cell, remove seed point

Integration loop, for each seed point **x**:

- interpolate ***v*** trilinearly to local coordinates (ξ, η, ζ)
- do an integration step, producing a new point **x'**
- incremental point location: For position **x'** find cell and local coordinates (ξ', η', ζ') making use of information (coordinates, local coordinates, cell) of old point **x**

Termination criteria:

- grid boundary reached
- step count limit reached
- optional: velocity close to zero
- optional: time limit reached
- optional: arc length limit reached

Integration step: widely used integration methods:

- Euler (used only in special speed-optimized techniques, e.g. GPU-based texture advection)

$$\mathbf{x}_{new} = \mathbf{x} + \mathbf{v}(\mathbf{x}, t) \cdot \triangle t$$

- Runge-Kutta, 2nd or 4th order

Higher order than 4th?

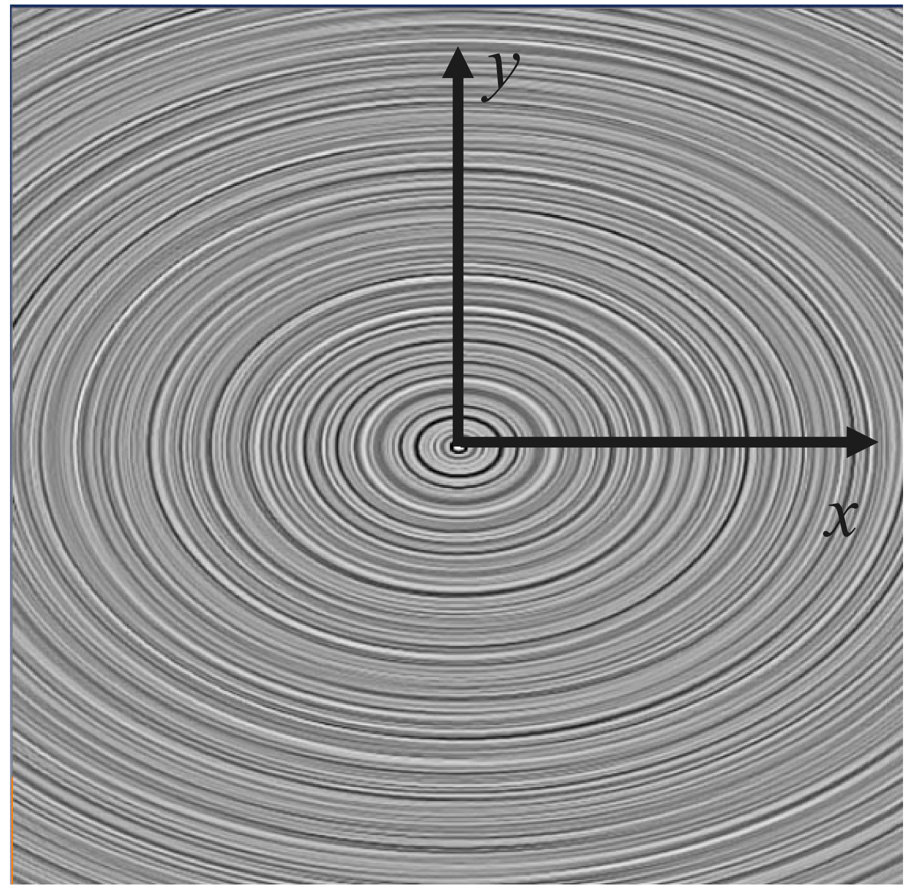
- often too slow for visualization
- study (Yeung/Pope 1987) shows that, when using standard trilinear interpolation, interpolation errors dominate integration errors.

Numerical Integration

- **Numerical integration of stream lines:**

- approximate streamline by polygon $\mathbf{x}_i$

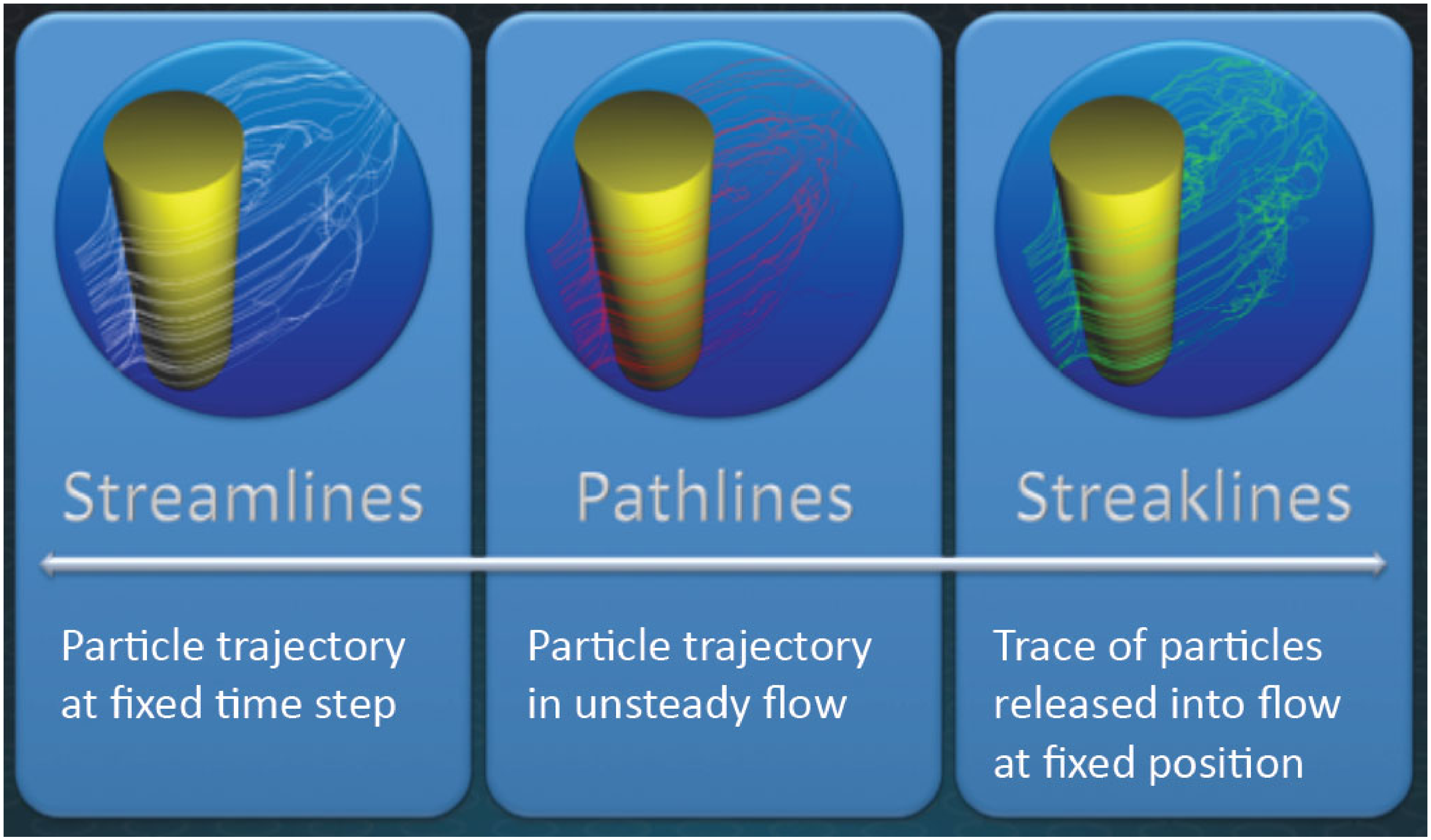
- **Testing example:**
  - $\mathbf{v}(x,y) = (-y, x/2)^T$
  - exact solution: ellipses
  - starting integration from (0,-1)



See slides in lecture 23!

Integral Curves, Pt. 2

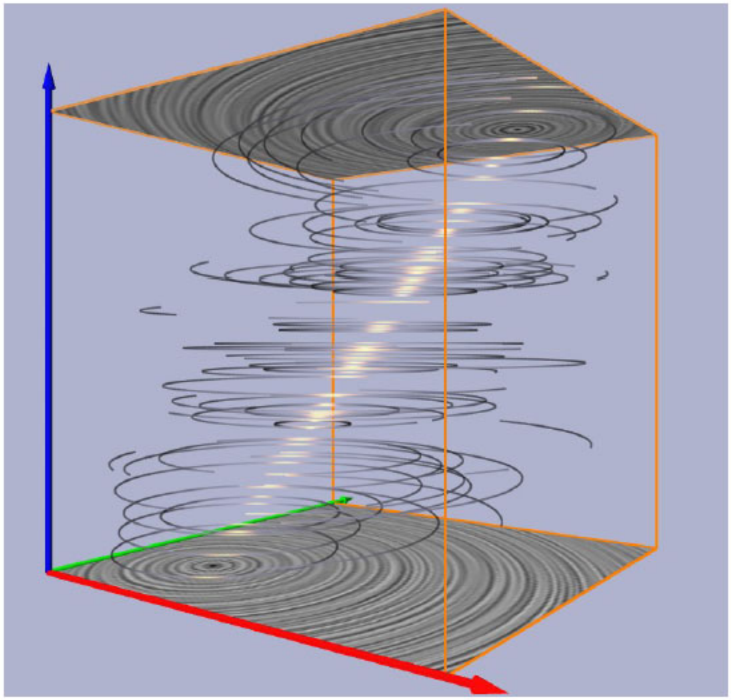
Integral Curves



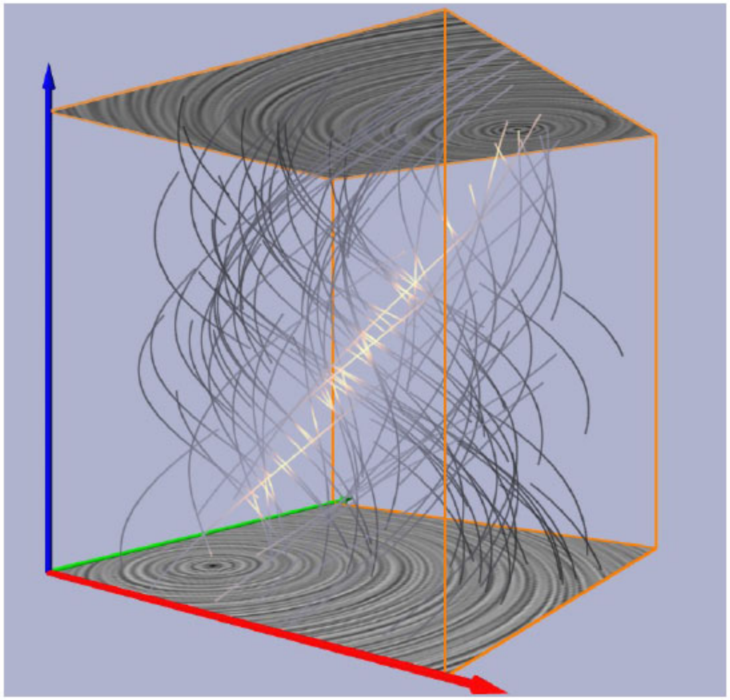
Stream Lines vs. Path Lines Viewed Over Time

Plotted with time as third dimension

- Tangent curves to a (n + 1)-dimensional vector field



Stream Lines



Path Lines

Streamline

- Curve parallel to the vector field in each point for a fixed time

Pathline

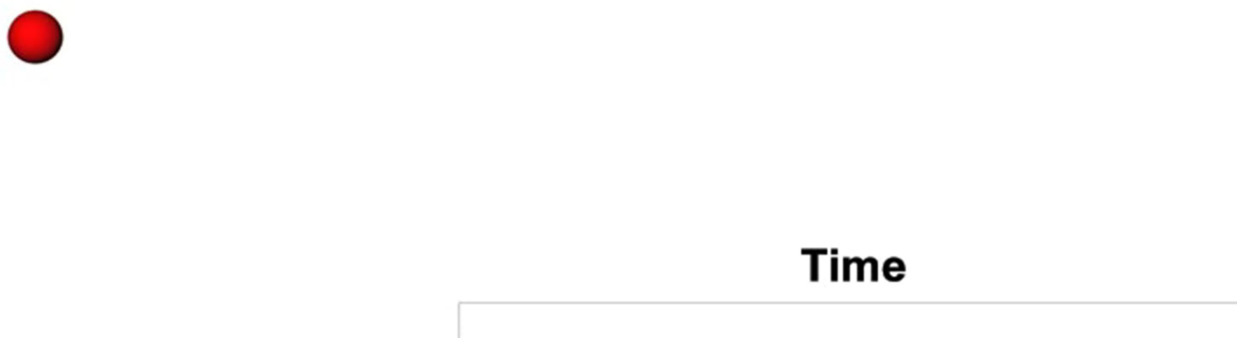
- Describes motion of a massless particle over time

Streakline

- Location of all particles released at a *fixed position* over time

Timeline

- Location of all particles released along a line at a *fixed time*



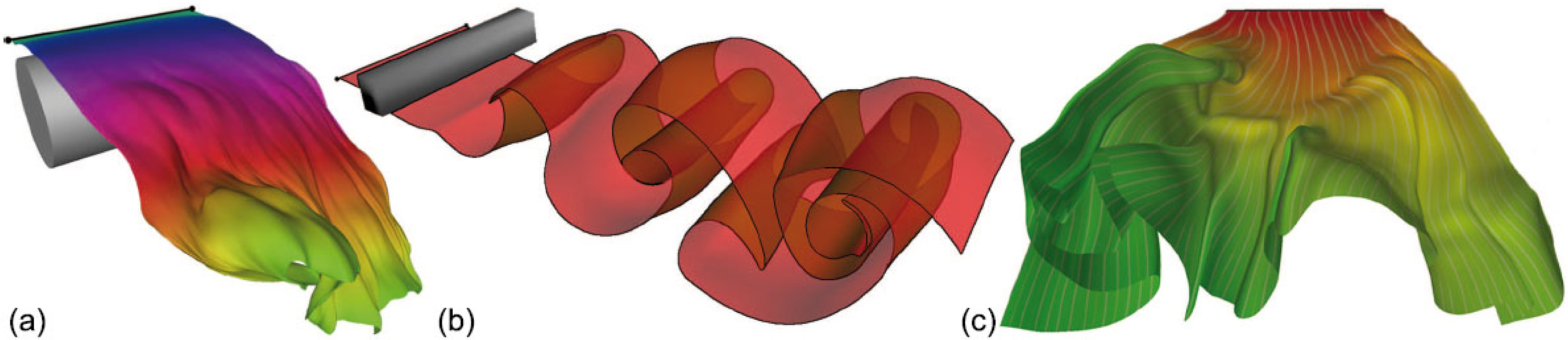
streak line

location of all particles set out at a fixed point at different times

Surfaces Instead of Lines

Seeding from a line instead of from a point

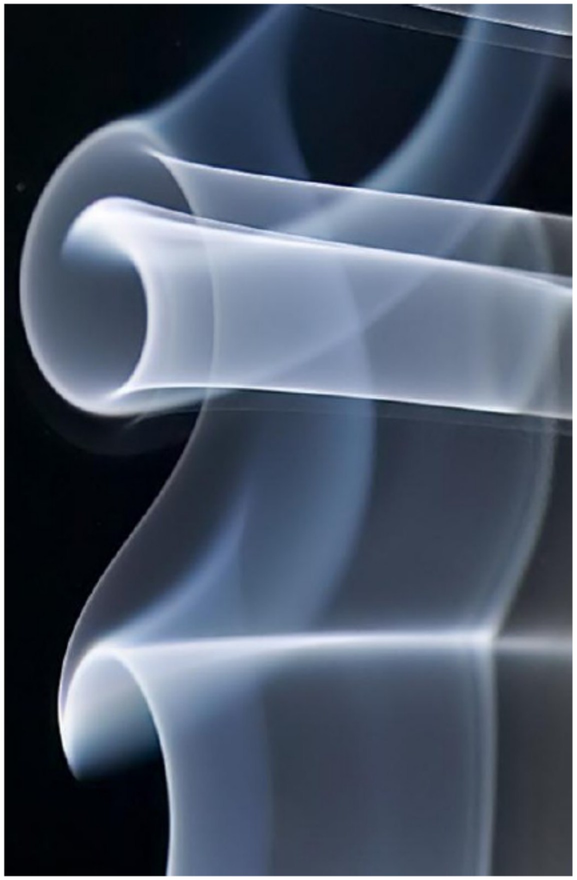
Example: streak surfaces



Volumes: seeding from a surface instead of a line

Real “Streak Surfaces”

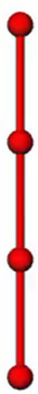
Artistic photographs of smoke

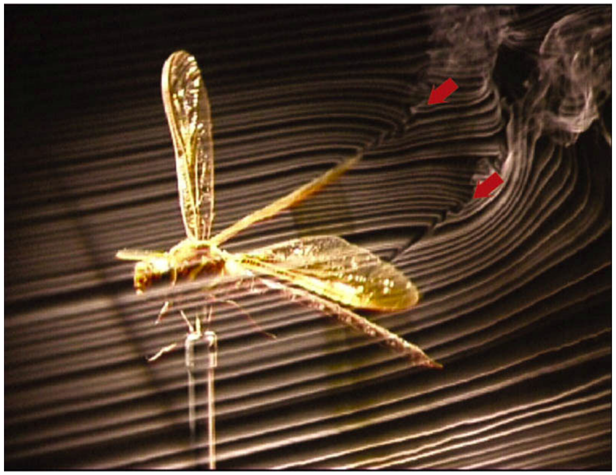


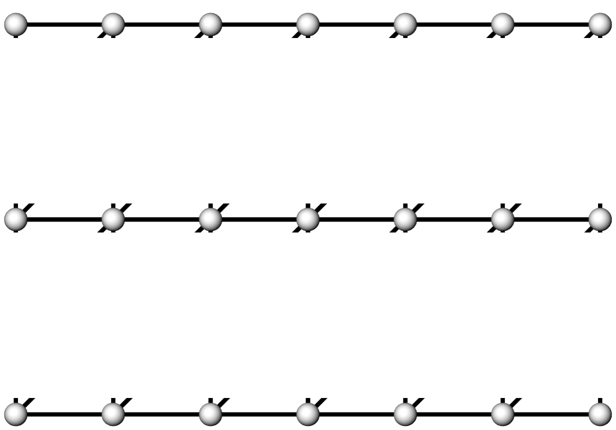
Time

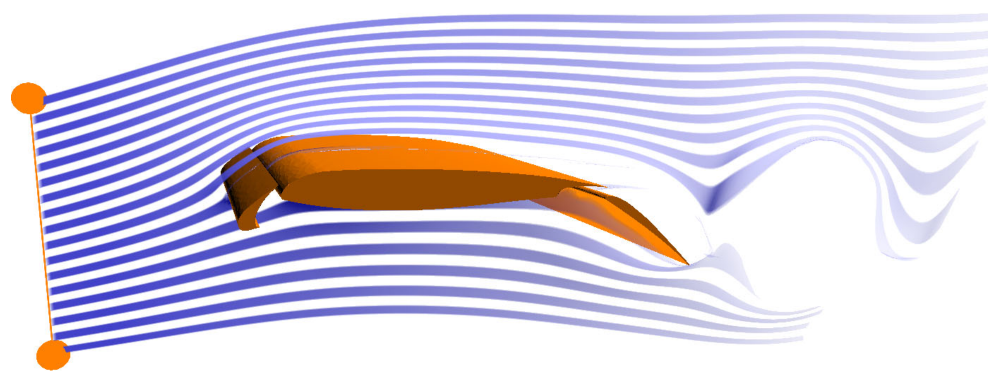
streak line

streak surface





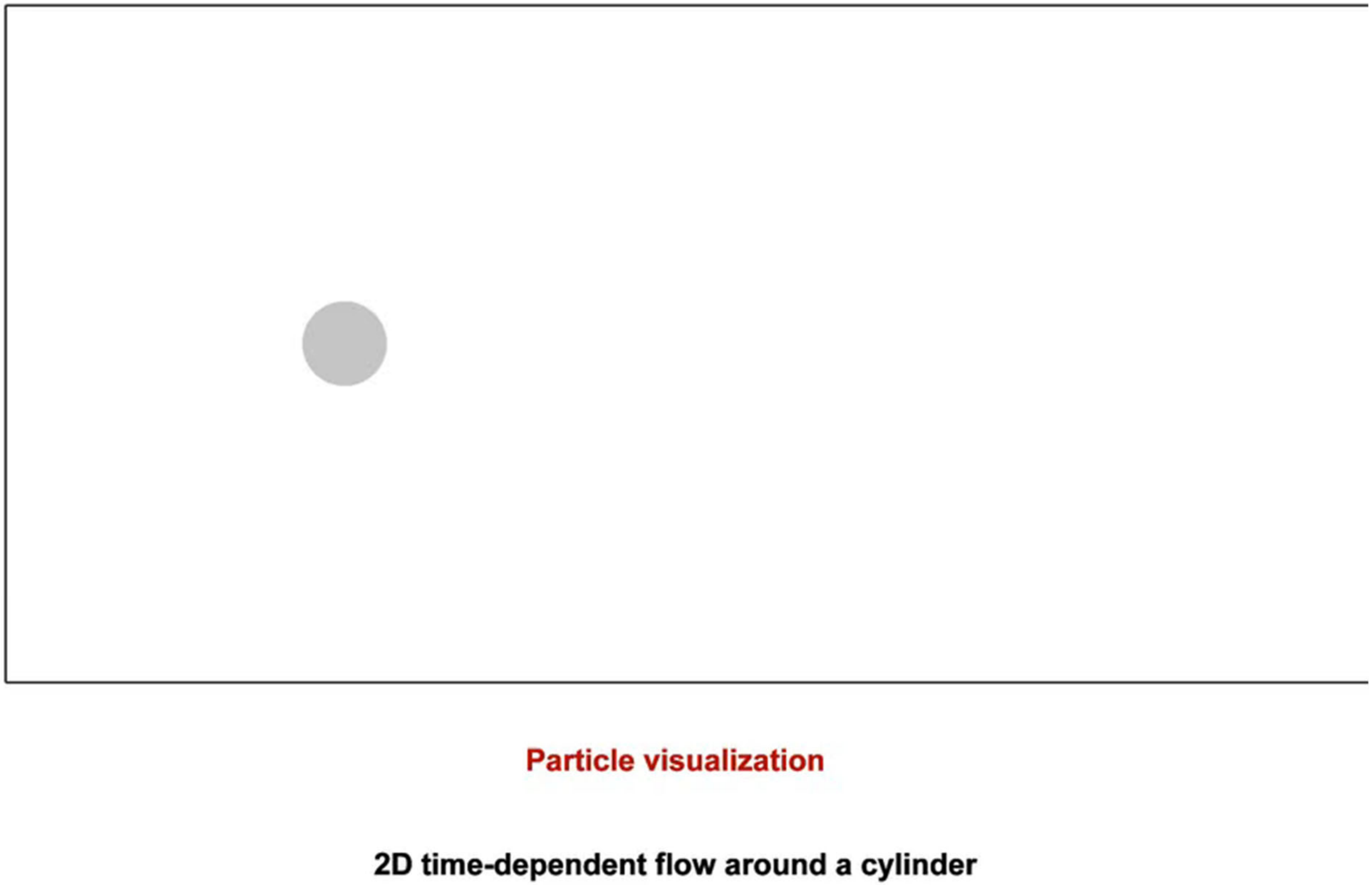




[Data courtesy of Günther (TU Berlin)]

fixed zero opacity rows

break connectivity

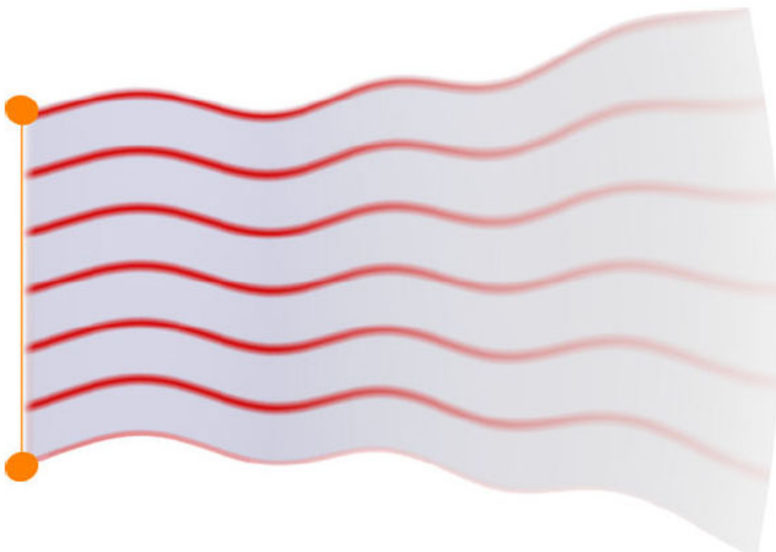


time line

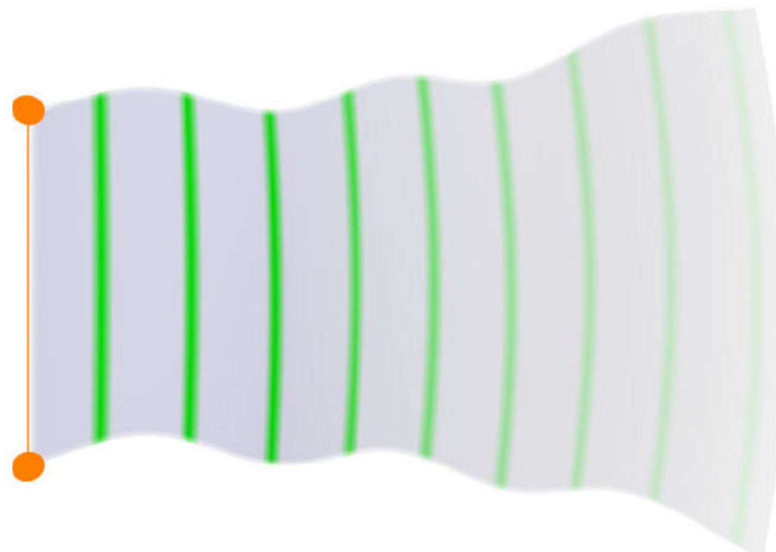
location of all particles set out on a certain line at a fixed time

Streak Lines vs. Time Lines

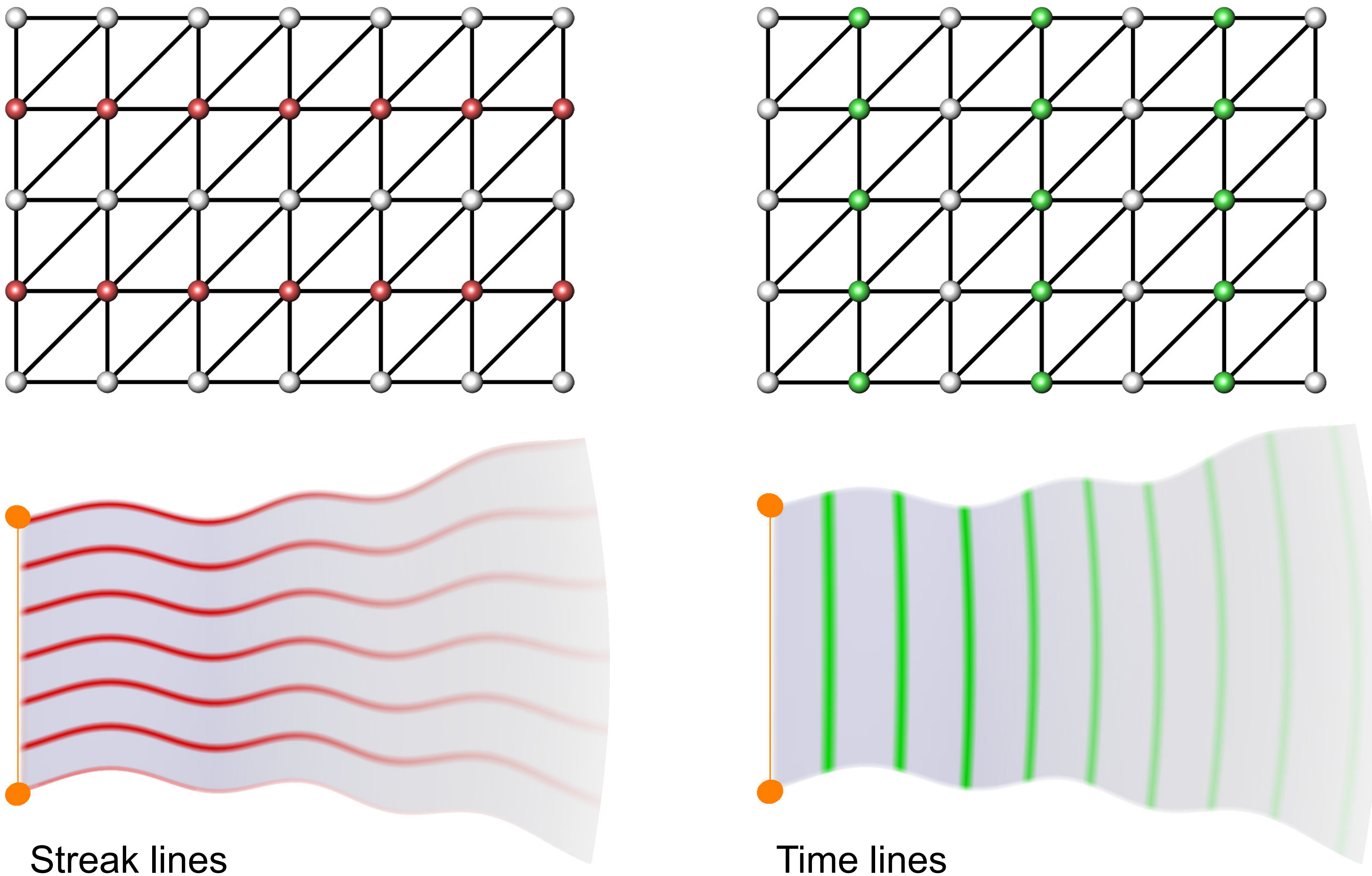
(on a streak surface)



Streak Lines



Time Lines


Streak lines
Time lines

The Flow / Flow Map of a Vector Field (1)

Flow of a *steady (time-independent)* vector field

- Map source position x "forward" ($t>0$) or "backward" ($t<0$) by time t

$$\boxed{\phi(x,t)} \qquad \boxed{\phi_t(x)}$$

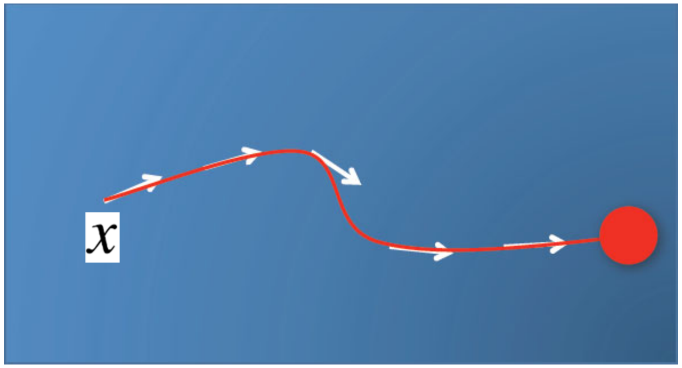
with

$$\phi_0(x) = x$$

$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

$$\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad (x,t) \mapsto \phi(x,t).$$

$$\phi_t : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \phi_t(x).$$



The Flow / Flow Map of a Vector Field (1)

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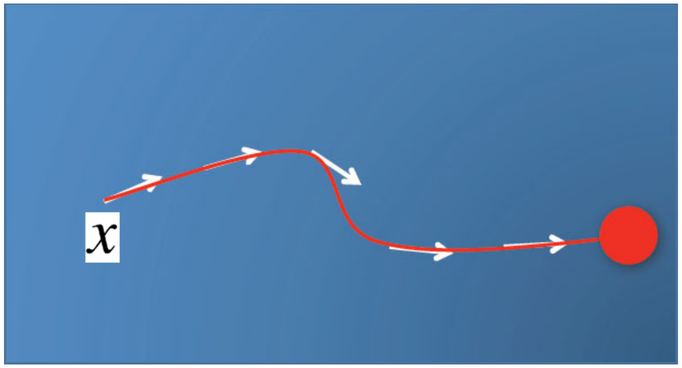
$$\boxed{\phi(x,t)} \qquad \boxed{\phi_t(x)} \qquad \text{with} \quad \phi_0(x) = x, \quad \phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi(x,t). \qquad \phi_t : M \to M, \quad x \mapsto \phi_t(x).$$



The Flow / Flow Map of a Vector Field (1)

Flow of a *steady (time-independent)* vector field

- Map source position *x* “forward” (*t*>*0*) or “backward” (*t*<*0*) by time *t*

$$\boxed{\phi(x,t)} \qquad \boxed{\phi_t(x)}$$

with

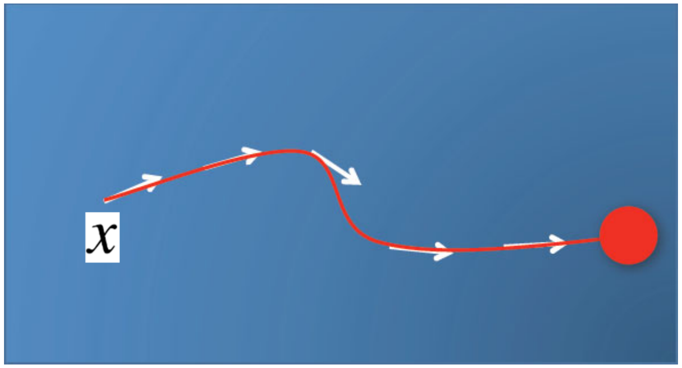
$$\phi_0(x) = x$$
$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi(x,t).$$

$$\phi_t : M \to M, \quad x \mapsto \phi_t(x).$$

$$\phi(x,t) = x + \int_0^t \mathbf{v}(\phi(x,\tau))\, \mathrm{d}\tau$$

(on a general manifold *M*, integration is performed in coordinate charts)



The Flow / Flow Map of a Vector Field (1)

Flow of a *steady (time-independent)* vector field

- Map source position *x* “forward” (*t*>*0*) or “backward” (*t*<*0*) by time *t*

$\boxed{\phi(x,t)}$ $\boxed{\phi_t(x)}$ with $\phi_0(x) = x$

$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$

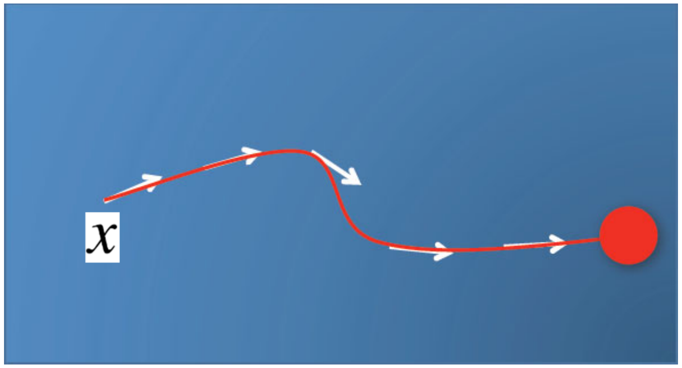
$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi(x,t).$$

$$\phi_t : M \to M, \quad x \mapsto \phi_t(x).$$

- Unsteady flow? Just fix arbitrary time T

$$\phi(x,t) = x + \int_0^t \mathbf{v}(\phi(x,\tau),\mathrm{T})\,\mathrm{d}\tau$$

(on a general manifold *M*, integration is performed in coordinate charts)



The Flow / Flow Map of a Vector Field (1)

Flow of a *steady (time-independent)* vector field

- Map source position *x* “forward” (*t*>*0*) or “backward” (*t*<*0*) by time *t*

$$\boxed{\phi(x,t)} \qquad \boxed{\phi_t(x)} \qquad \text{with} \quad \phi_0(x) = x, \quad \phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

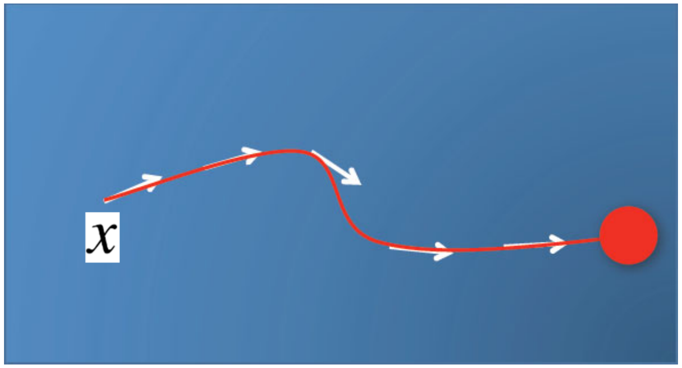
$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi(x,t).$$

$$\phi_t : M \to M, \quad x \mapsto \phi_t(x).$$

Can write explicitly as function of independent variable *t*, with *position x fixed*

$$t \mapsto \phi(x,t) \qquad t \mapsto \phi_t(x)$$

= stream line going through point *x*



The Flow / Flow Map of a Vector Field (2)

Flow of an *unsteady (time-dependent)* vector field

- Map source position *x* from time *s* to destination position at time *t*
  (t < s is allowed: map forward or backward in time)

$$\boxed{\psi_{t,s}(x)}$$

with

$$\psi_{t,s}(x) = x + \int_s^t \mathbf{v}\big(\psi_{\tau,s}(x), \tau\big)\, \mathrm{d}\tau$$

$$\psi_{s,s}(x) = x$$

$$\psi_{t,r}\big(\psi_{r,s}(x)\big) = \psi_{t,s}(x)$$

The Flow / Flow Map of a Vector Field (3)

Flow of an *unsteady (time-dependent)* vector field

- Map source position *x* from time *s* to destination position at time *t* (t < s is allowed: map forward or backward in time)

$$\boxed{\psi_{t,s}(x)} \qquad \psi_{t,s}(x) = x + \int_s^t \mathbf{v}\big(\psi_{\tau,s}(x), \tau\big)\,\mathrm{d}\tau$$

Can write explicitly as function of t, *with s and x fixed*

$t \mapsto \psi_{t,s}(x)$ → path line

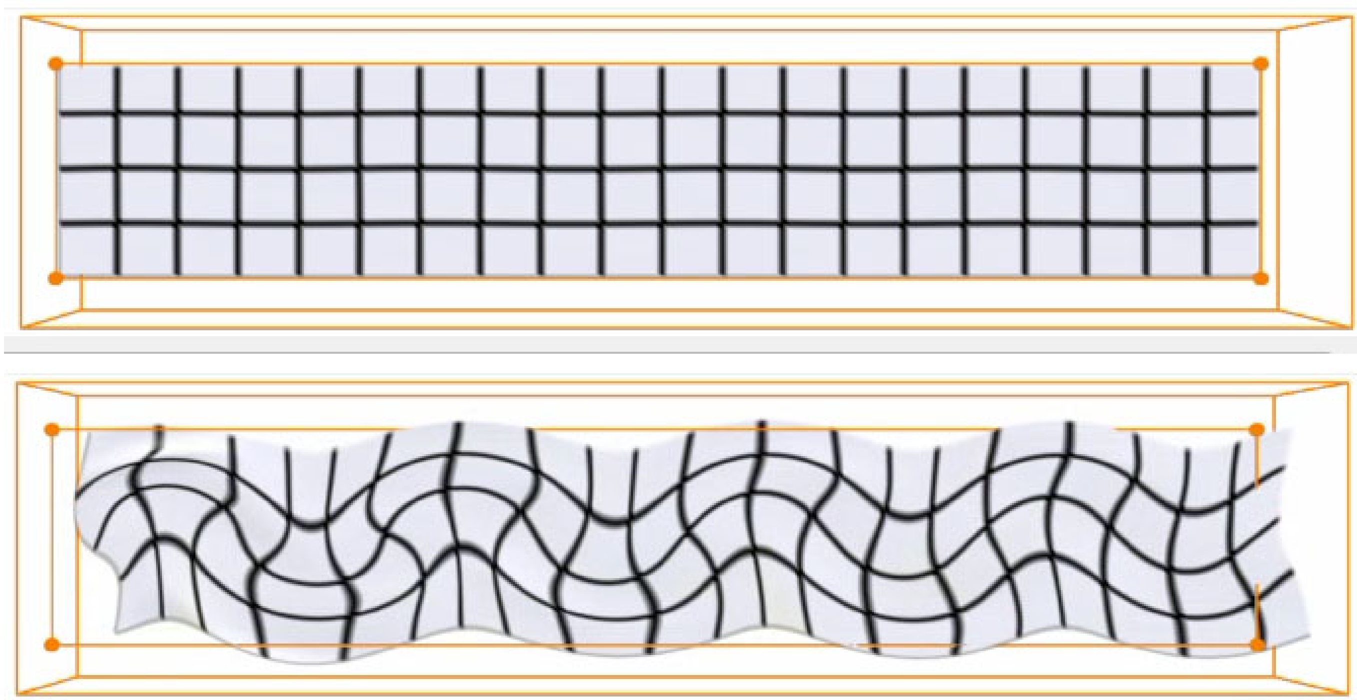
Can write explicitly as function of s, *with t and x fixed*

$s \mapsto \psi_{t,s}(x)$ → streak line

$\psi_{t,s}(x)$ is also often written as ***flow map*** $\phi_t^\tau(x)$ (with t:=s and either τ:=t or τ:=t-s)

The Flow / Flow Map of a Vector Field (4)

Can map a whole set of points (or the entire domain) through the flow map (this map is a *diffeomorphism*): $t \mapsto \psi_{t,s}(U)$



U

$= \psi_{s,s}(U)$

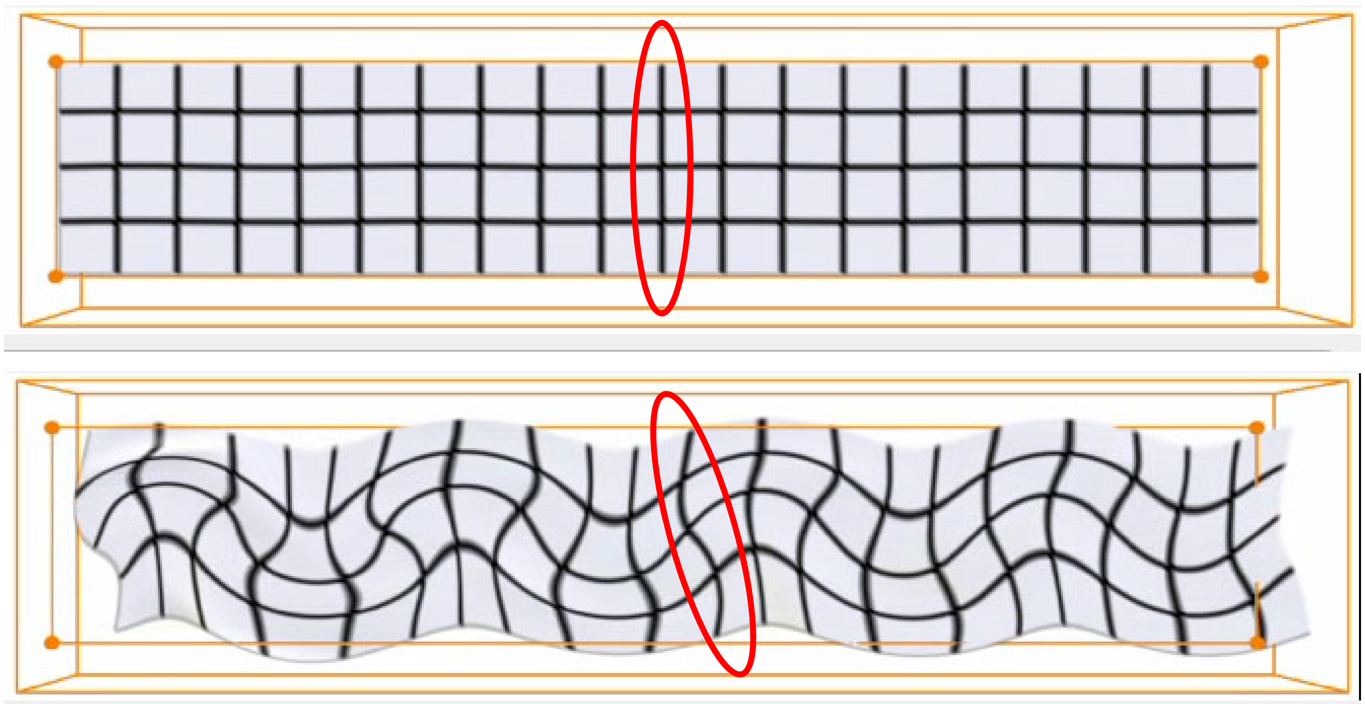
$\psi_{t,s}(U)$

(this is a *time surface!*)

The Flow / Flow Map of a Vector Field (5)

Time line: Map a whole curve from one fixed time (s) to another time (t)

$$t \mapsto \psi_{t,s}(c(\lambda))$$



$c(\lambda)$

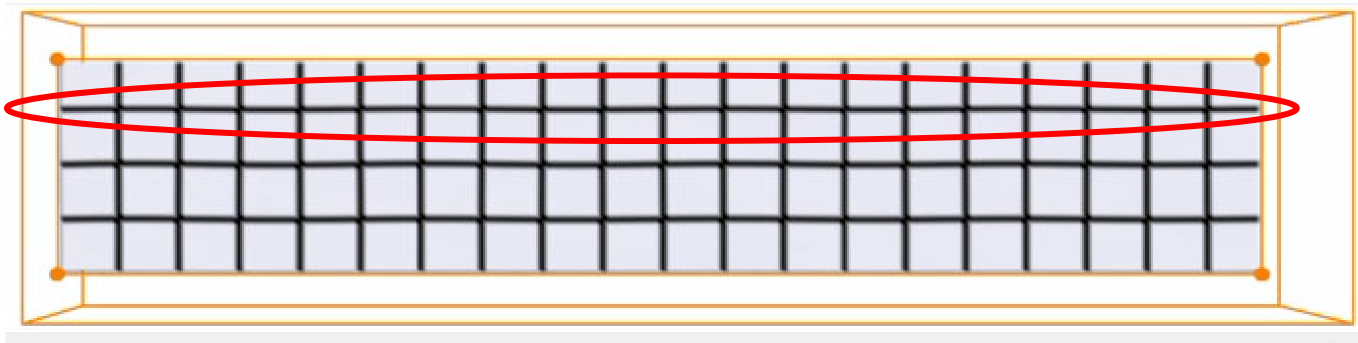
$= \psi_{s,s}(c(\lambda))$

$\psi_{t,s}(c(\lambda))$

The Flow / Flow Map of a Vector Field (5)

Time line: Map a whole curve from one fixed time (s) to another time (t)

$$t \mapsto \psi_{t,s}(c(\lambda))$$



$c(\lambda)$

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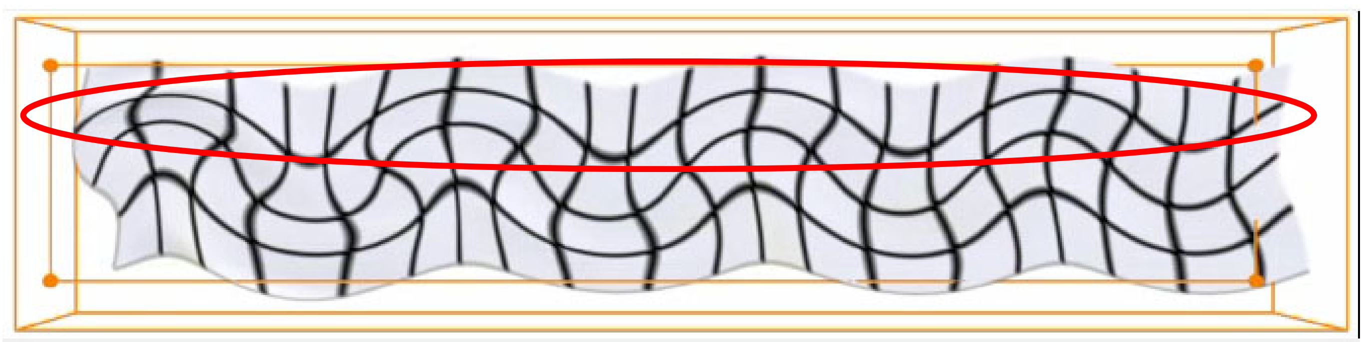


$\psi_{t,s}(c(\lambda))$

Streamline

- Curve parallel to the vector field in each point for a fixed time

Pathline

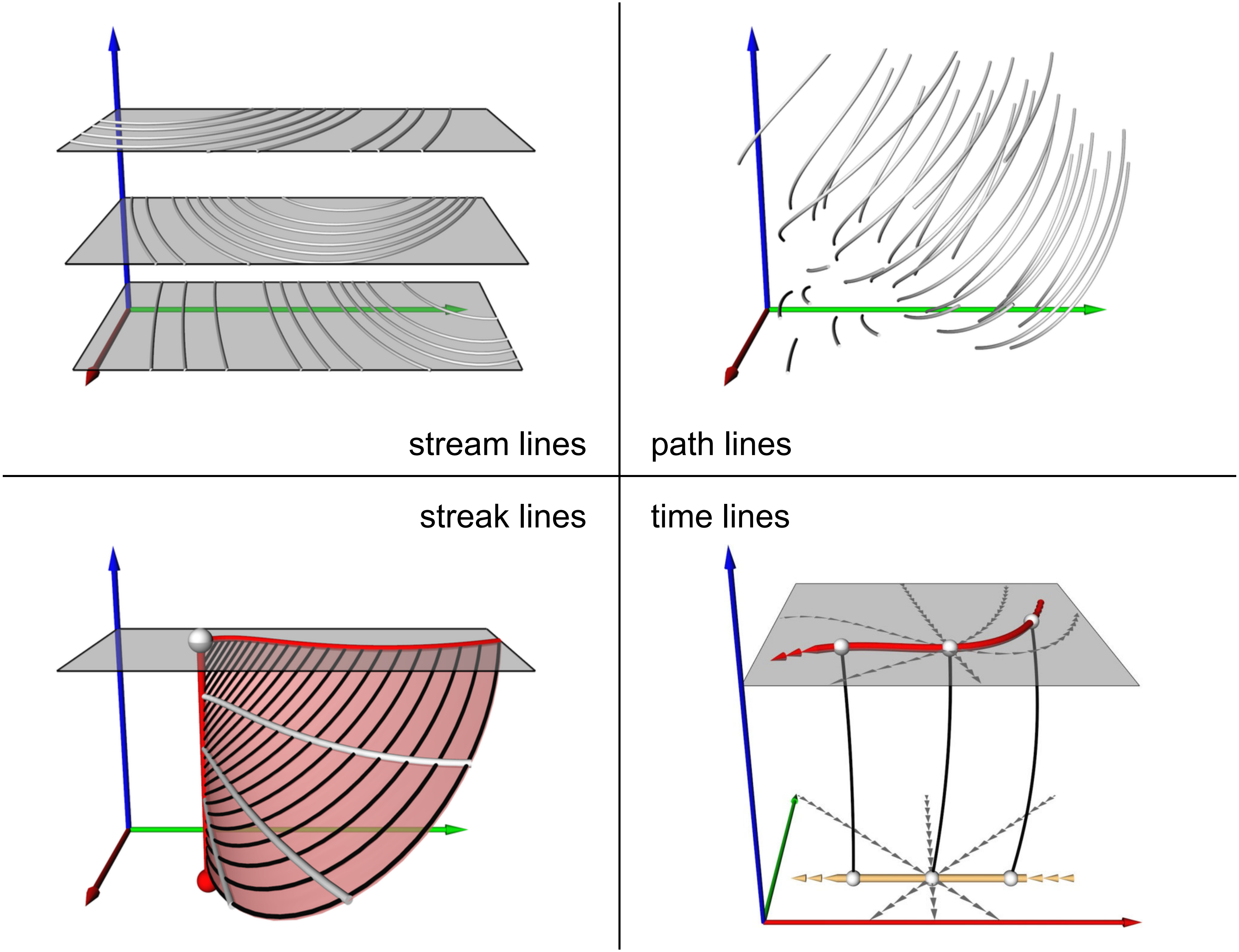
- Describes motion of a massless particle over time

Streakline

- Location of all particles released at a *fixed position* over time

Timeline

- Location of all particles released along a line at a *fixed time*


stream lines
path lines
streak lines
time lines

Comparison of techniques:

(1) Pathlines:
- are physically meaningful
- allow comparison with experiment (observe marked particles)
- are well suited for dynamic visualization (of particles)

(2) Streamlines:
- are only geometrically, not physically meaningful
- are easiest to compute (no temporal interpolation, single IVP)
- are better suited for static visualization (prints)
- don't intersect (under reasonable assumptions)

(3) Streaklines:

- are physically meaningful
- allow comparison with experiment (dye injection)
- are well suited for static and dynamic visualization
- good choice for fast moving vortices
- can be approximated by set of disconnected particles

(4) Timelines:

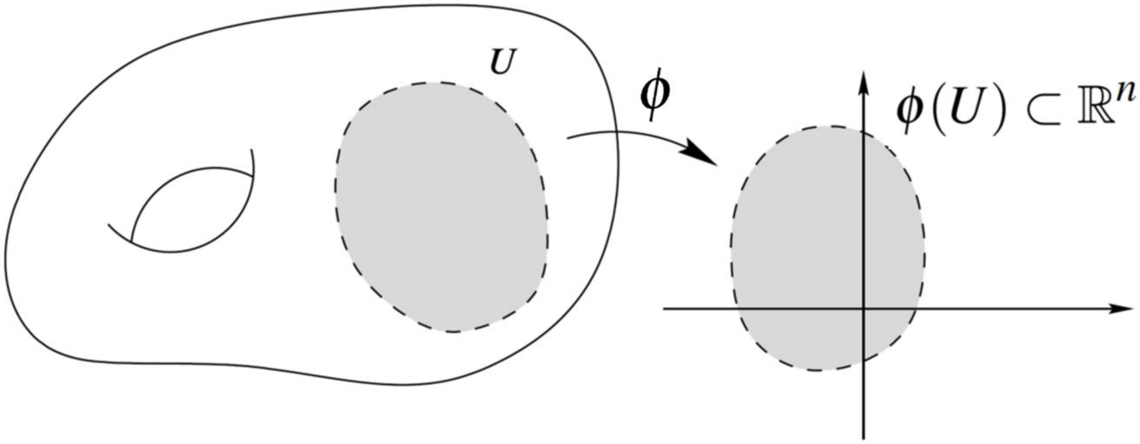
- are physically meaningful
- are well suited for static and dynamic visualization
- can be approximated by set of disconnected particles

Bonus Slides:
Vector Fields on General Manifolds, Coordinate Charts

Interlude: Coordinate Charts

Coordinate chart

$$\phi : U \subset M \to \mathbb{R}^n,$$
$$x \mapsto (x^1, x^2, \dots, x^n).$$



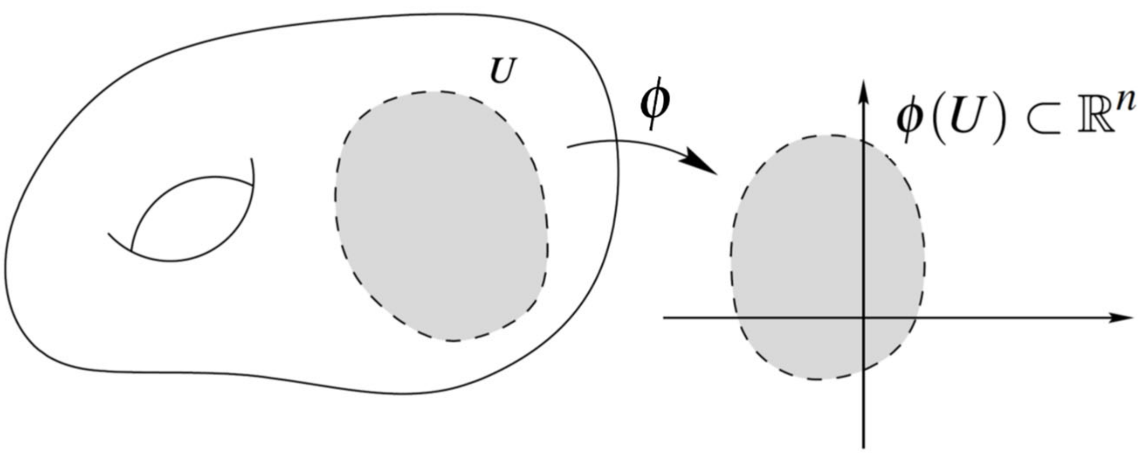
Interlude: Coordinate Charts

Coordinate chart

$$\phi : U \subset M \to \mathbb{R}^n,$$
$$x \mapsto (x^1, x^2, \ldots, x^n).$$

Coordinate functions

$$x^i : U \subset M \to \mathbb{R},$$
$$x \mapsto x^i(x).$$



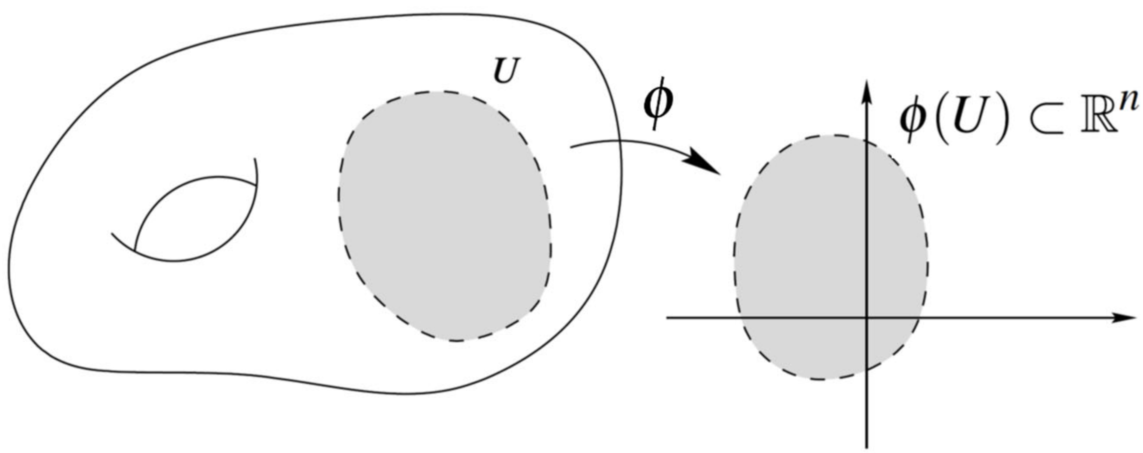
Interlude: Coordinate Charts

Coordinate charts

$$\phi_\alpha : U_\alpha \subset M \to \mathbb{R}^n,$$
$$x \mapsto (x^1, x^2, \dots, x^n).$$

Atlas

$$\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$$



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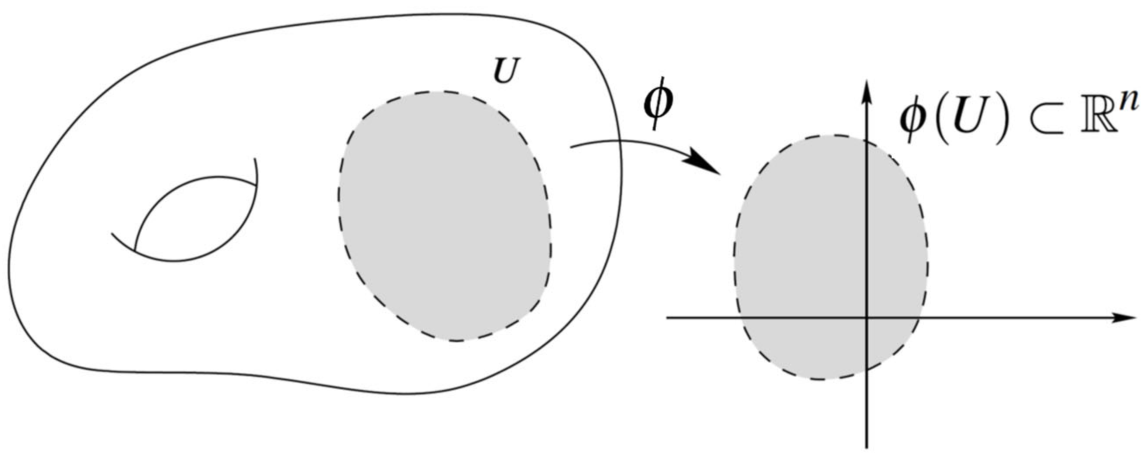
$$\phi_\alpha : U_\alpha \subset M \to \mathbb{R}^n,$$
$$x \mapsto (x^1(x), x^2(x), \ldots, x^n(x)).$$

Atlas

$$\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$$



Vector Fields vs. Vectors in Components

Because Euclidean space is most common, often slightly sloppy notation

$$\mathbf{v} : U \subset \mathbb{R}^2 \to \mathbb{R}^2,$$
$$(x,y) \mapsto \begin{bmatrix} u \\ v \end{bmatrix}.$$

Vector Fields vs. Vectors in Components

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Vector Fields vs. Vectors in Components

Because Euclidean space is most common, often slightly sloppy notation

$$\mathbf{v}: U \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto \begin{bmatrix} u \\ v \end{bmatrix}.$$

$$\mathbf{v}: U \subset \mathbb{R}^3 \to \mathbb{R}^3, \quad (x,y,z) \mapsto \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

$$\mathbf{v}: U \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$$

$$\mathbf{v}: U \subset \mathbb{R}^3 \to \mathbb{R}^3, \quad (x,y,z) \mapsto \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}.$$

Vector Fields vs. Vectors in Components

$$\mathbf{v} : U \subset \mathbb{R}^n \to \mathbb{R}^n,$$

$$(x^1, x^2, \ldots, x^n) \mapsto \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}.$$

$$\mathbf{v} : U \subset \mathbb{R}^n \to \mathbb{R}^n,$$

$$(x^1, x^2, \ldots, x^n) \mapsto \begin{pmatrix} v^1(x^1, x^2, \ldots, x^n) \\ v^2(x^1, x^2, \ldots, x^n) \\ \vdots \\ v^n(x^1, x^2, \ldots, x^n) \end{pmatrix}.$$

Vector Fields vs. Vectors in Components

$$\mathbf{v}: U \subset \mathbb{R}^n \to \mathbb{R}^n,$$

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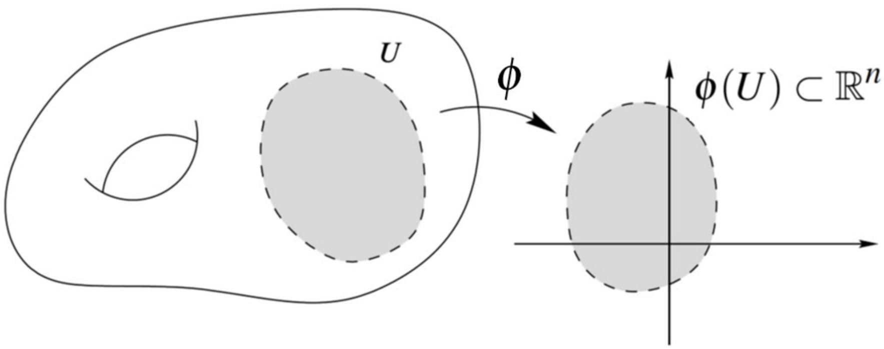

$$\mathbf{v}: U \subset \mathbb{R}^n \to \mathbb{R}^n,$$

$$(x^1, x^2, \ldots, x^n) \mapsto \begin{pmatrix} v^1(x^1, x^2, \ldots, x^n) \\ v^2(x^1, x^2, \ldots, x^n) \\ \vdots \\ v^n(x^1, x^2, \ldots, x^n) \end{pmatrix}.$$

$$\mathbf{v}\big|_U: \phi(U) \subset \mathbb{R}^n \to \mathbb{R}^n,$$

$$(x^1, x^2, \ldots, x^n) \mapsto \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}.$$

Vector Fields vs. Vectors in Components

Need basis vector fields

$$\mathbf{e}_i : U \subset M \to TM, \quad x \mapsto \mathbf{e}_i(x)$$

$$\{\mathbf{e}_i(x)\}_{i=1}^{n} \quad \text{basis for } T_xM$$

Vector Fields vs. Vectors in Components

Need basis vector fields

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Coordinate basis:

$$\mathbf{e}_i := \frac{\partial}{\partial x^i}$$

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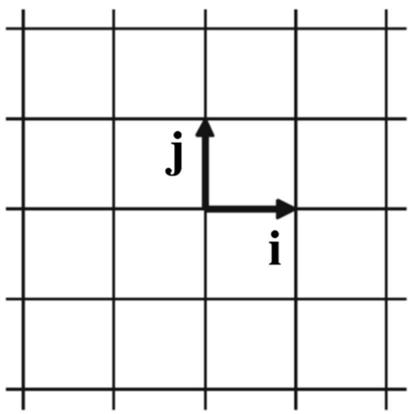
Examples of Coordinate Curves and Bases

Coordinate functions, coordinate curves, bases

- Coordinate functions are real-valued ("scalar") functions on the domain
- On each coordinate curve, *one* coordinate changes, *all others stay constant*
- Basis: n linearly independent vectors at each point of domain

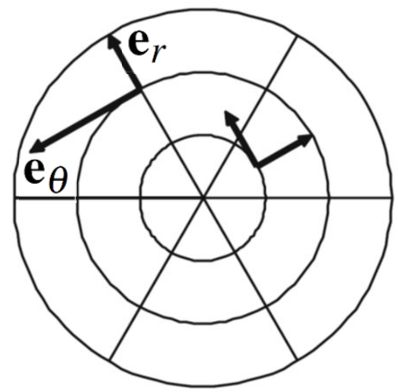
Cartesian coordinates

$$x^1 = x$$
$$x^2 = y$$
$$\mathbf{e}_1 = \frac{\partial}{\partial x} = \mathbf{i}$$
$$\mathbf{e}_2 = \frac{\partial}{\partial y} = \mathbf{j}$$



polar coordinates

$$x^1 = r$$
$$x^2 = \theta$$
$$\mathbf{e}_1 = \frac{\partial}{\partial r} = \mathbf{e}_r$$
$$\mathbf{e}_2 = \frac{\partial}{\partial \theta} = \mathbf{e}_\theta$$



Bonus Slides:
Vectors as Derivative Operators

Vectors as Derivative Operators

A vector applied to a (real) function on the manifold gives the *directional derivative* in that direction

- From this viewpoint, the vector is a derivative operator (actually, a *derivation*)
- Can be used as *definition* of a vector (must fulfill props. of a derivation; esp. Leibniz rule)

$$f : M \to \mathbb{R}, \qquad \mathbf{v} f$$
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Kronecker delta ("identity matrix")

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For vector field: obtain directional derivative at each point

$$\mathbf{v}f : M \to \mathbb{R}, \quad x \mapsto \mathbf{v}(x)\,f = df(\mathbf{v}(x)).$$

remember that this just *looks* scary (maybe) ...

Thank you.

Thanks for material

- Helwig Hauser
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- Ronny Peikert
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