CS 247 – Scientific Visualization
Lecture 23: Vector / Flow Visualization, Pt. 2

Markus Hadwiger, KAUST

Reading Assignment #12 (until Apr 30)

Read (required):

- Data Visualization book
  - Chapter 6 (Vector Visualization)
    - Beginning (before 6.1)
    - Chapters 6.2, 6.3, 6.5
- More general vector field basics (the book is not very precise on the basics)

  `https://en.wikipedia.org/wiki/Vector_field`

Read (optional):

- Paper:
  Bruno Jobard and Wilfrid Lefer
  Creating Evenly-Spaced Streamlines of Arbitrary Density,

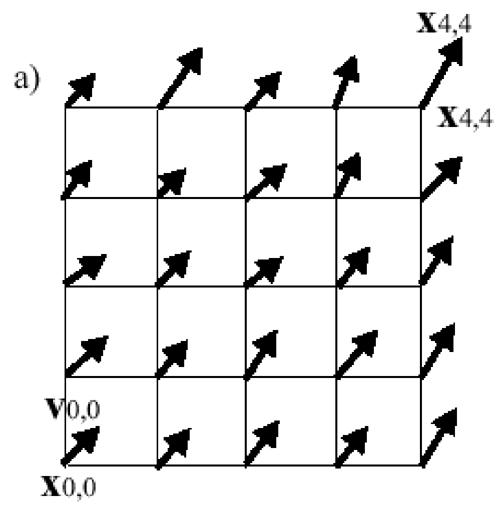
  `http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.29.9498`

Vector Fields

Each vector is usually thought of as a velocity vector

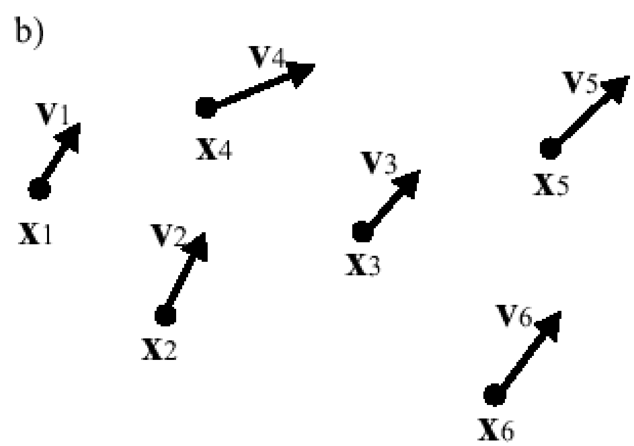
- Example for actual velocity: fluid flow
- But also force fields, etc. (e.g., electrostatic field)

Eulerian specification:



vectors given at grid points
(grid points do not move)

Lagrangian specification:



vectors given at particle positions
(particle positions do move)

Vector Fields

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- Example for actual velocity: fluid flow
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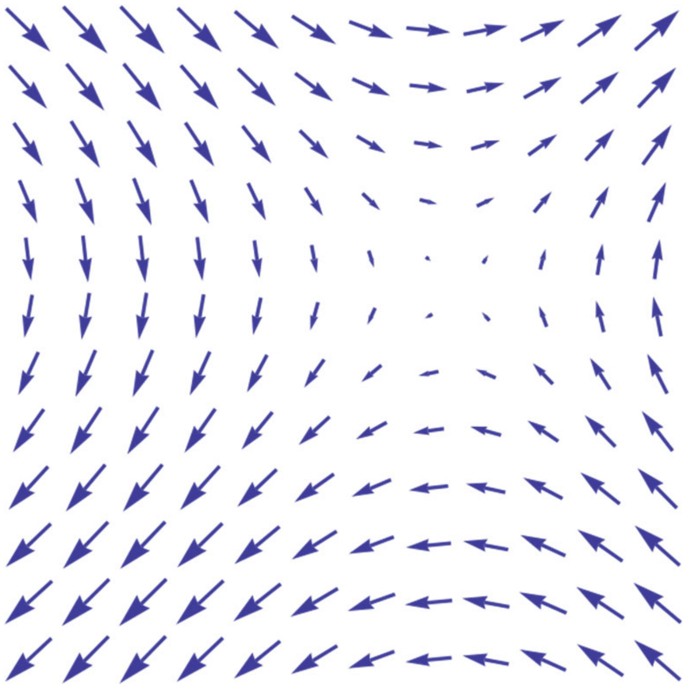
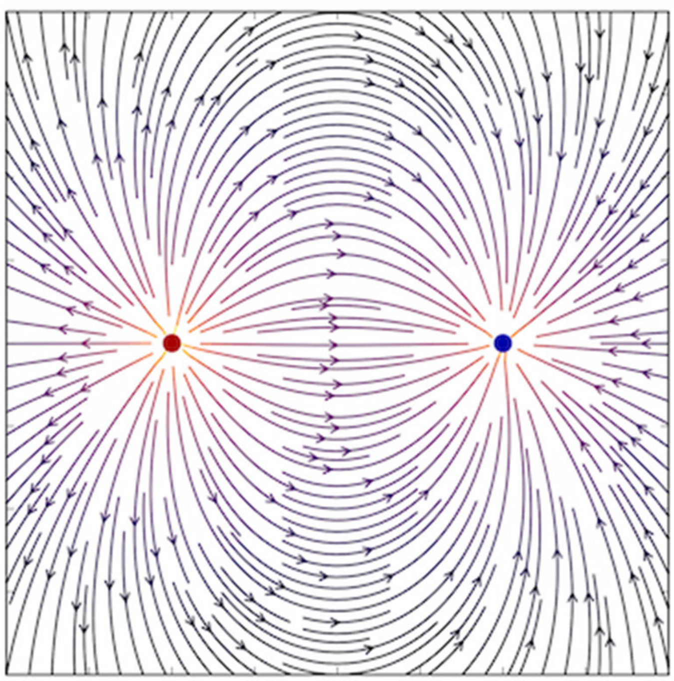
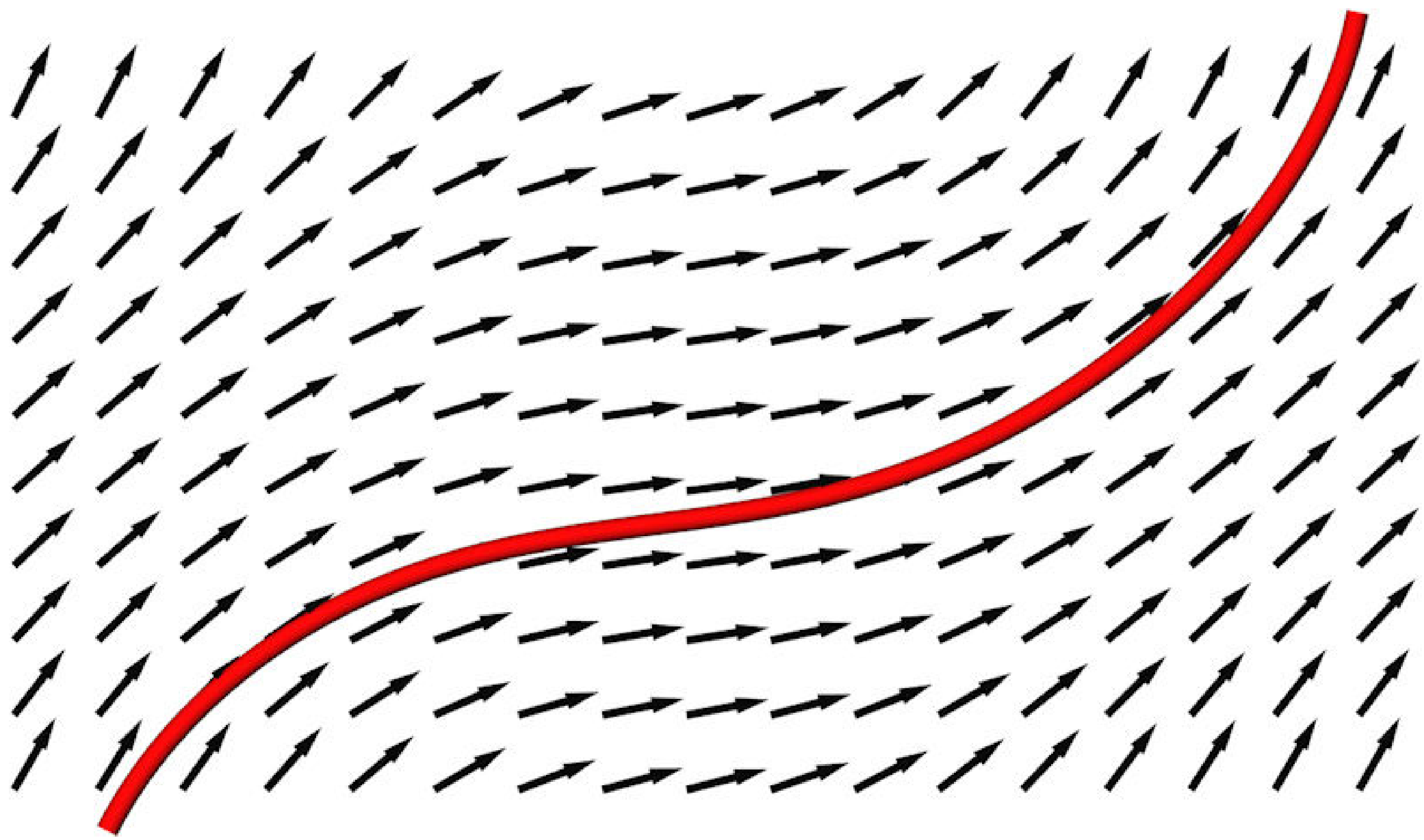




images from wikipedia

Integral Curves / Stream Objects

Integrating velocity over time yields spatial motion



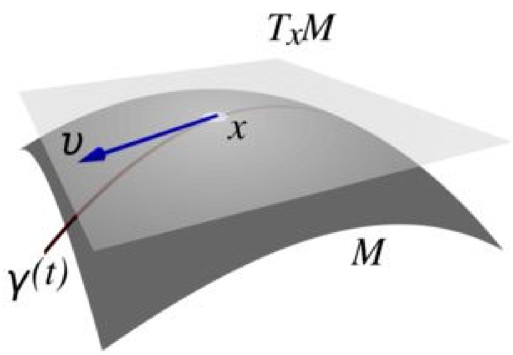
Vector Fields

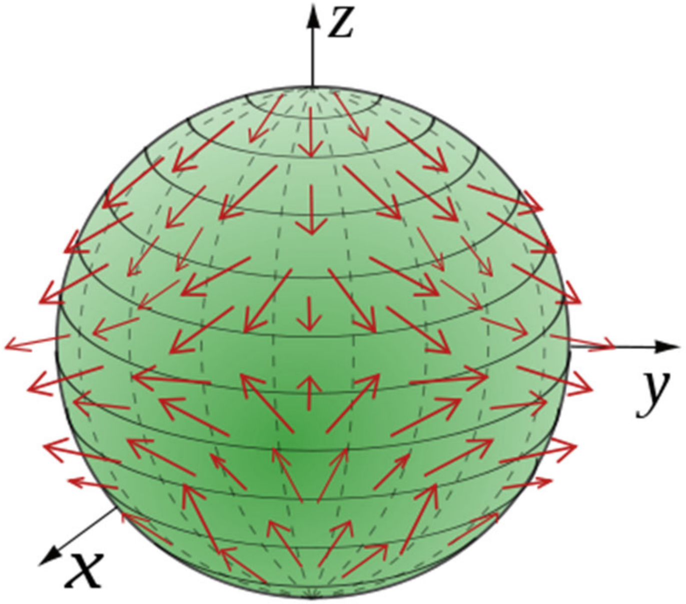
Each vector is usually thought of as a velocity vector

- Example for actual velocity: fluid flow
- But also force fields, etc. (e.g., electrostatic field)

Each vector in a vector field
lives in the **tangent space**
of the manifold at that point:

Each vector is a **tangent vector**





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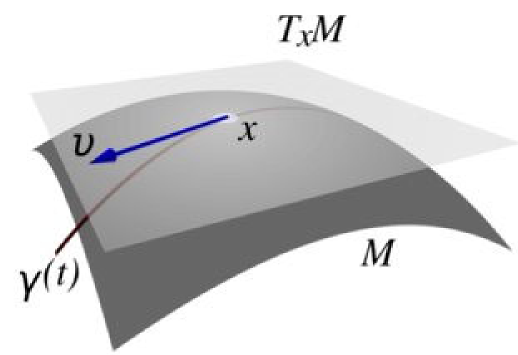
Vector Fields

Vector fields on general manifolds M (not just Euclidean space)

Tangent space at a point $x \in M$:

$$T_x M$$

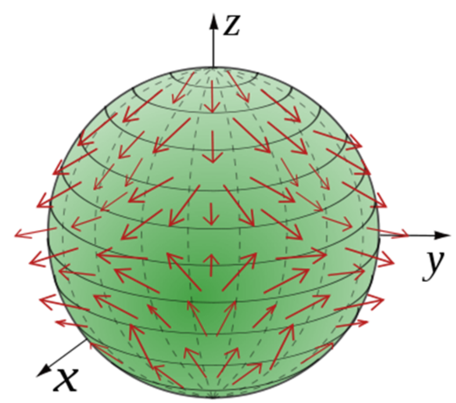


Tangent bundle: Manifold of all tangent spaces over base manifold

$$\pi : TM \to M$$

Vector field: *Section* of tangent bundle

$$s : M \to TM, \quad x \mapsto s(x). \qquad \pi(s(x)) = x$$



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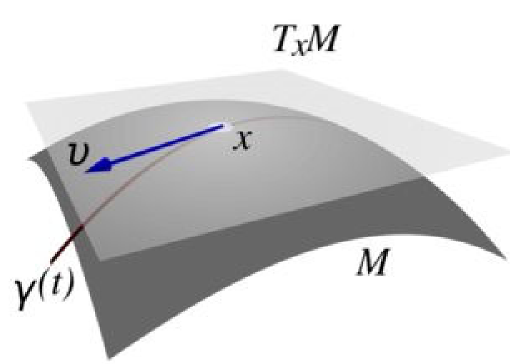

Tangent bundle: Manifold of all tangent spaces over base manifold

$$\pi : TM \rightarrow M$$

Vector field: *Section* of tangent bundle

$$\begin{aligned} \mathbf{v} : M &\rightarrow TM, \\ x &\mapsto \mathbf{v}(x). \end{aligned} \qquad \mathbf{v}(x) \in T_x M$$

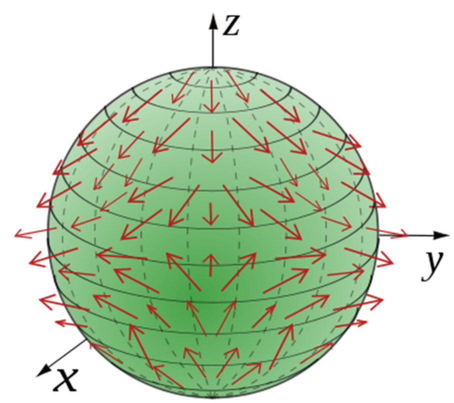

image from wikipedia

Vector fields

A static vector field $\mathbf{v}(\mathbf{x})$ is a vector-valued function of space.

A time-dependent vector field $\mathbf{v}(\mathbf{x},t)$ depends also on time.

In the case of velocity fields, the terms steady and unsteady flow are used.

The dimensions of **x** and **v** are equal, often 2 or 3, and we denote components by *x,y,z* and *u,v,w*:

$$\mathbf{x} = (x,y,z), \ \mathbf{v} = (u,v,w)$$

Sometimes a vector field is defined on a surface $\mathbf{x}(i,j)$. The vector field is then a function of parameters and time:

$$\mathbf{v}(i,j,t)$$

Steady vs. Unsteady Flow

- Steady flow: time-independent
  - Flow itself is static over time: $\mathbf{v}(\mathbf{x})$
  - Example: laminar flows

$$\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \mathbf{v}(x).$$

- Unsteady flow: time-dependent
  - Flow itself changes over time: $\mathbf{v}(\mathbf{x}, t)$
  - Example: turbulent flows

$$\mathbf{v} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad (x, t) \mapsto \mathbf{v}(x, t).$$

(here just for Euclidean domain; analogous on general manifolds)

Steady vs. Unsteady Flow

- Steady flow: time-independent
  - Flow itself is static over time: $\mathbf{v}(\mathbf{x})$
  - Example: laminar flows

$$\mathbf{v}:\ M \to \mathbb{R}^n,\quad x \mapsto \mathbf{v}(x).$$

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  - Example: turbulent flows

$$\mathbf{v}:\ M \times \mathbb{R} \to \mathbb{R}^n,\quad (x,t) \mapsto \mathbf{v}(x,t).$$

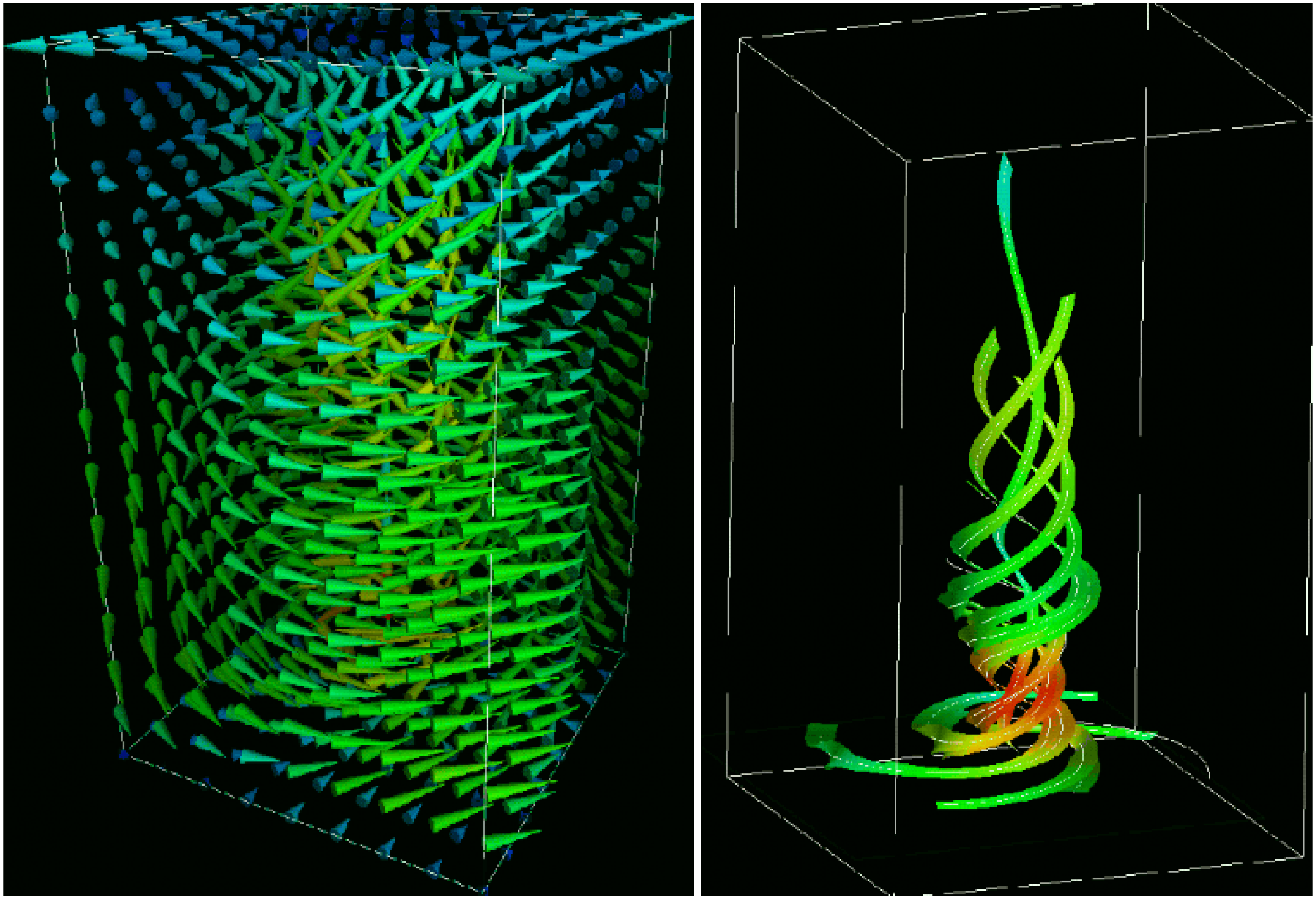
(here now for general manifolds)

Direct vs. Indirect Flow Visualization

- Direct flow visualization
  - Overview of current flow state
  - Visualization of vectors: arrow plots (“hedgehog” plots)

- Indirect flow visualization
  - Use intermediate representation: vector field integration over time
  - Visualization of temporal evolution
  - Integral curves: streamlines, pathlines, streaklines, timelines
  - Integral surfaces: streamsurfaces, pathsurfaces, streaksurfaces

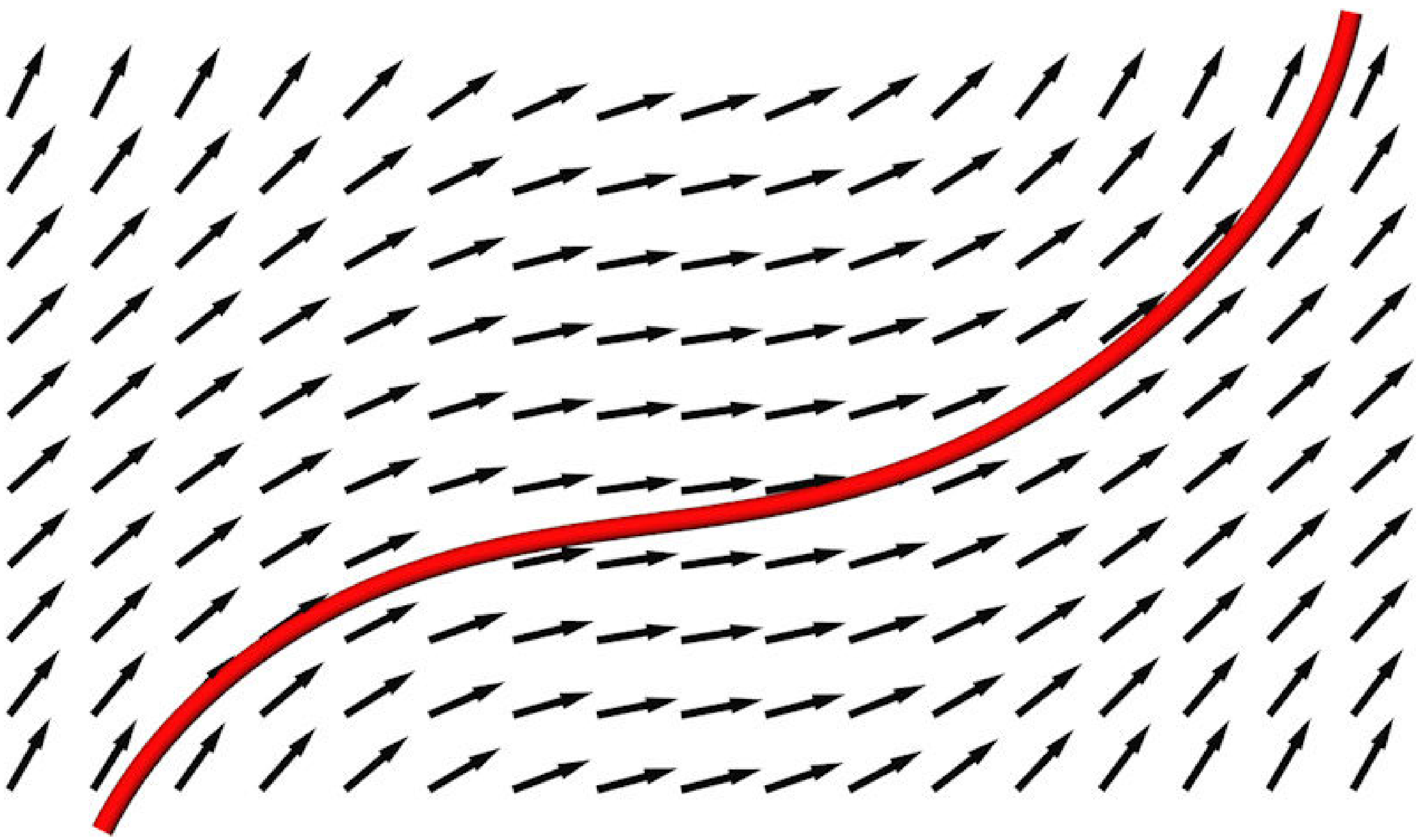
Direct vs. Indirect Flow Visualization



Integral Curves: Intro

Integral Curves / Stream Objects

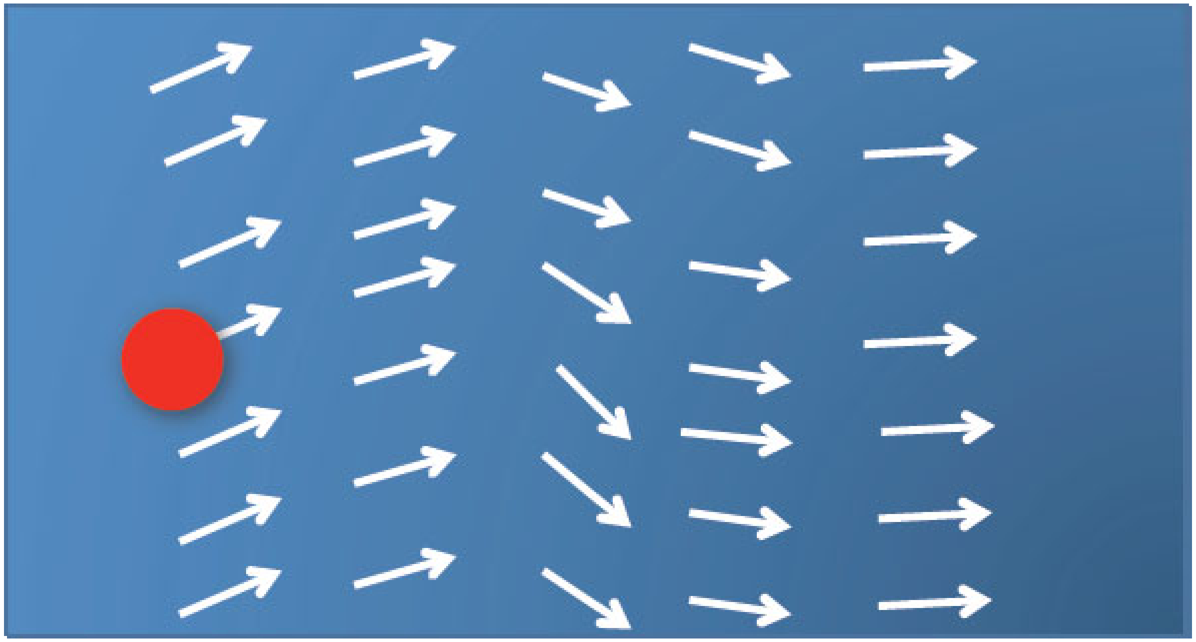
Integrating velocity over time yields spatial motion



Particle Trajectories

Courtesy Jens Krüger

Particle Trajectories



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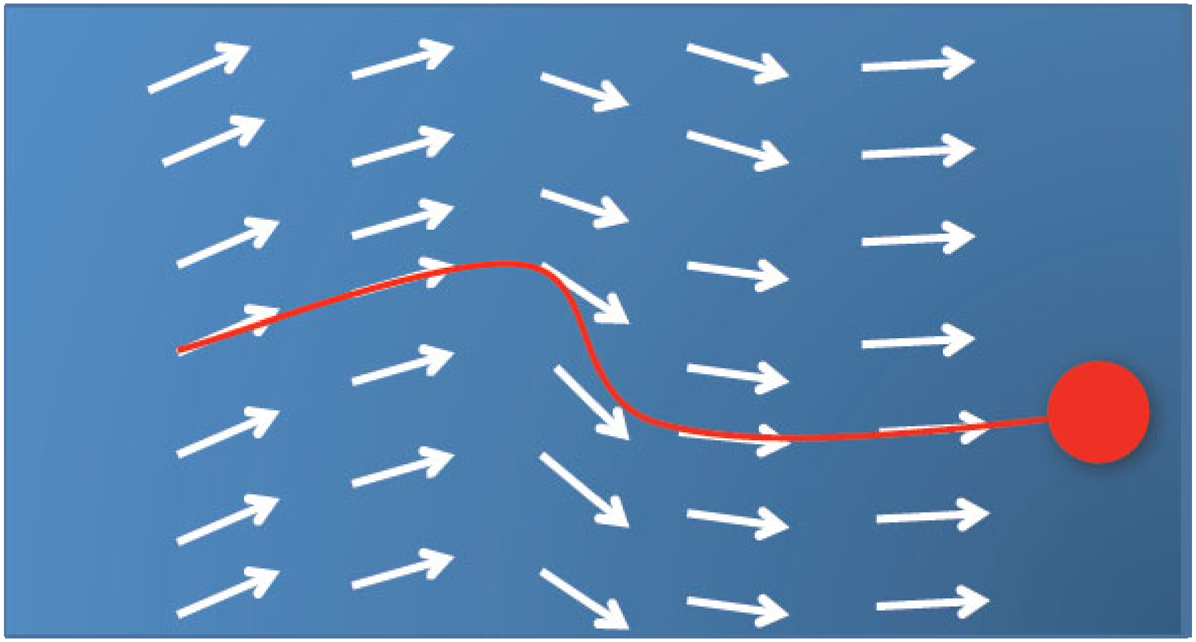
Particle Trajectories



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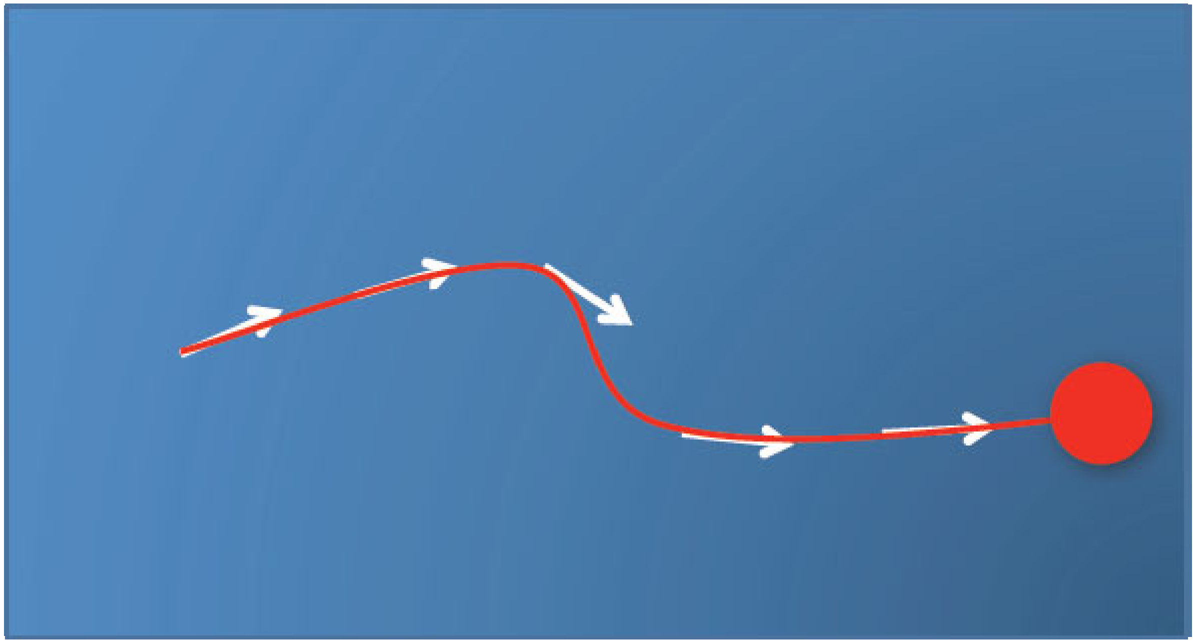
Particle Trajectories



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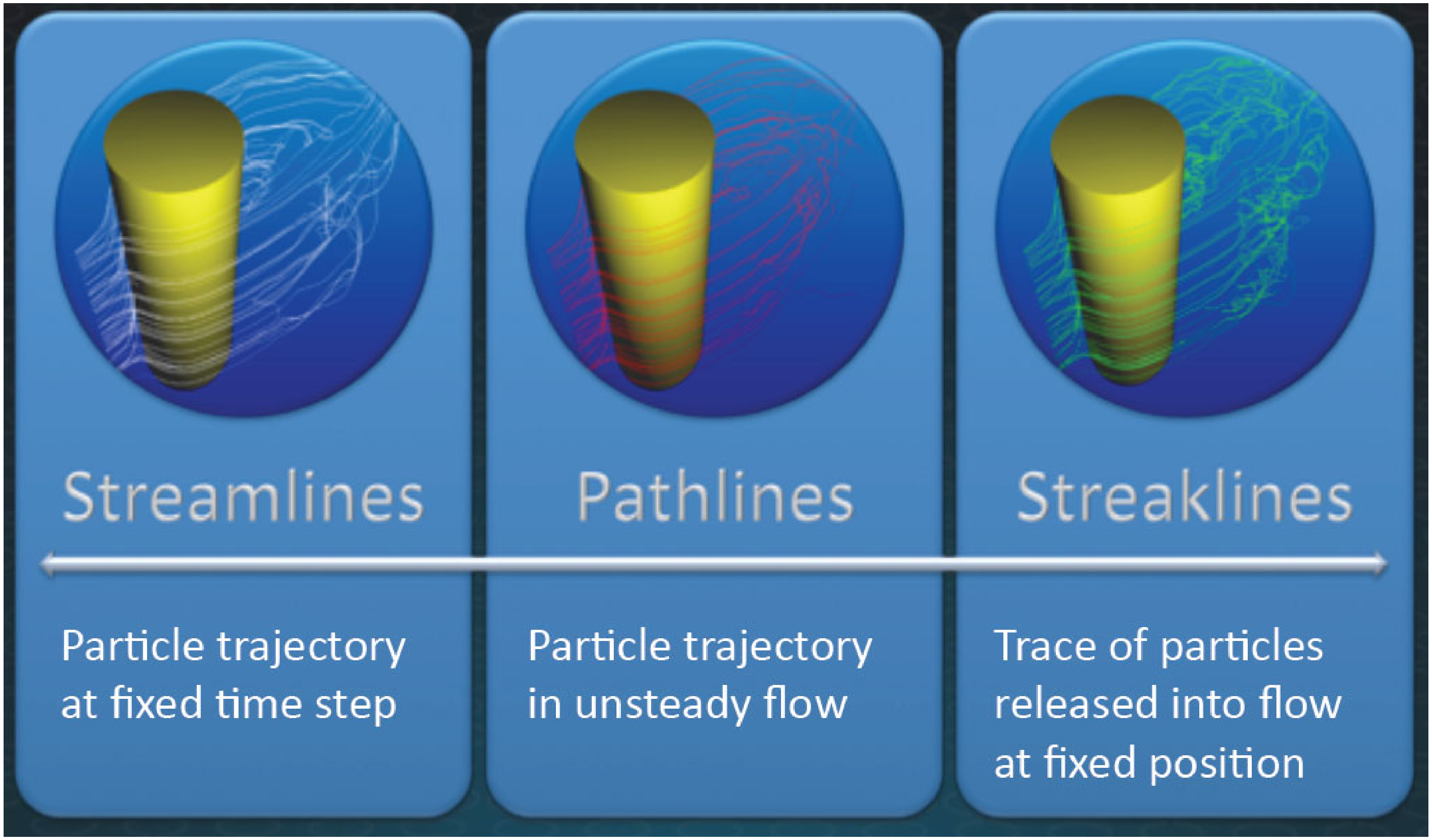
Integral Curves



Streamline

- Curve parallel to the vector field in each point for a fixed time

Pathline

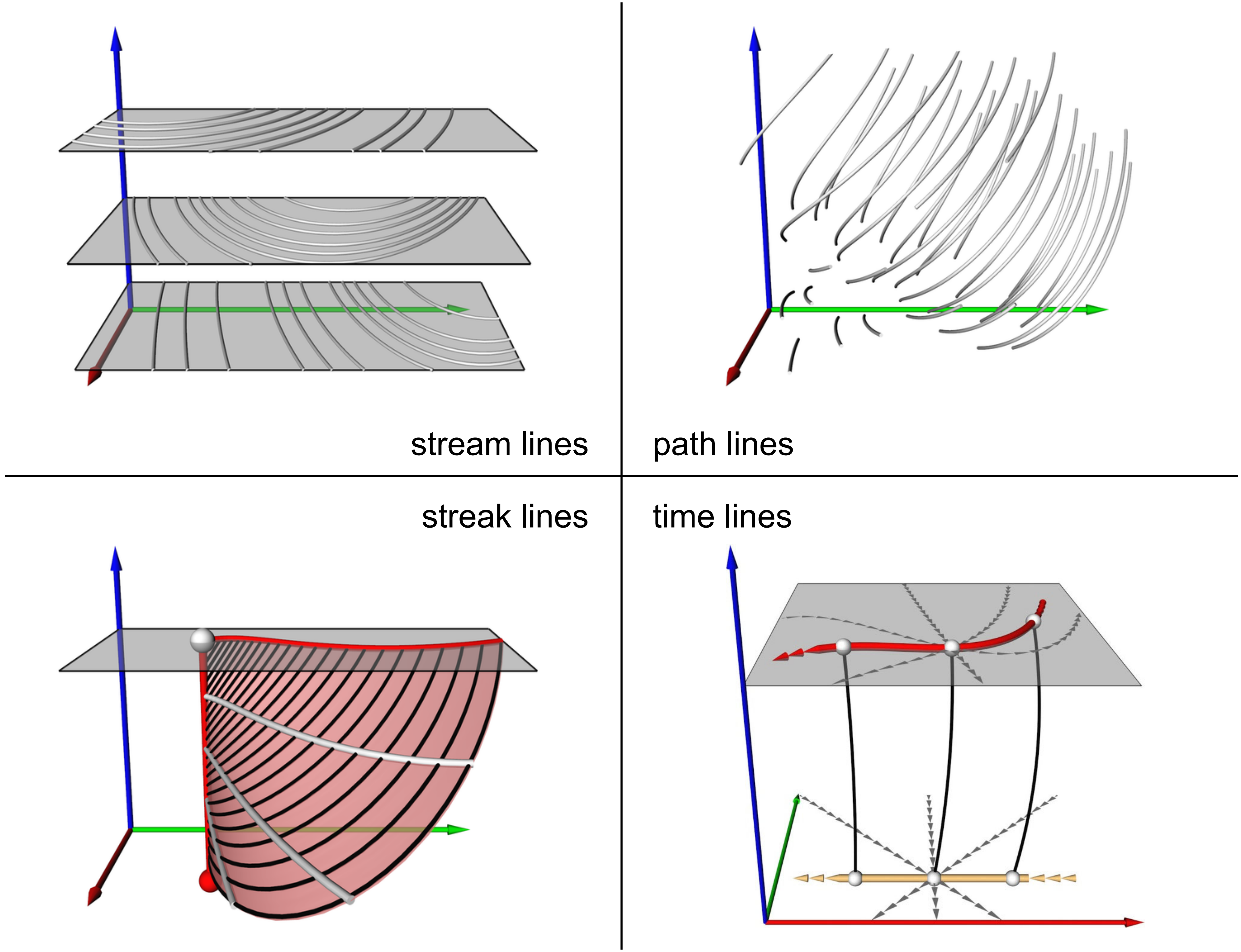
- Describes motion of a massless particle over time

Streakline

- Location of all particles released at a *fixed position* over time

Timeline

- Location of all particles released along a line at a *fixed time*


stream lines
path lines
streak lines
time lines

Comparison of techniques:

(1) Pathlines:
- are physically meaningful
- allow comparison with experiment (observe marked particles)
- are well suited for dynamic visualization (of particles)

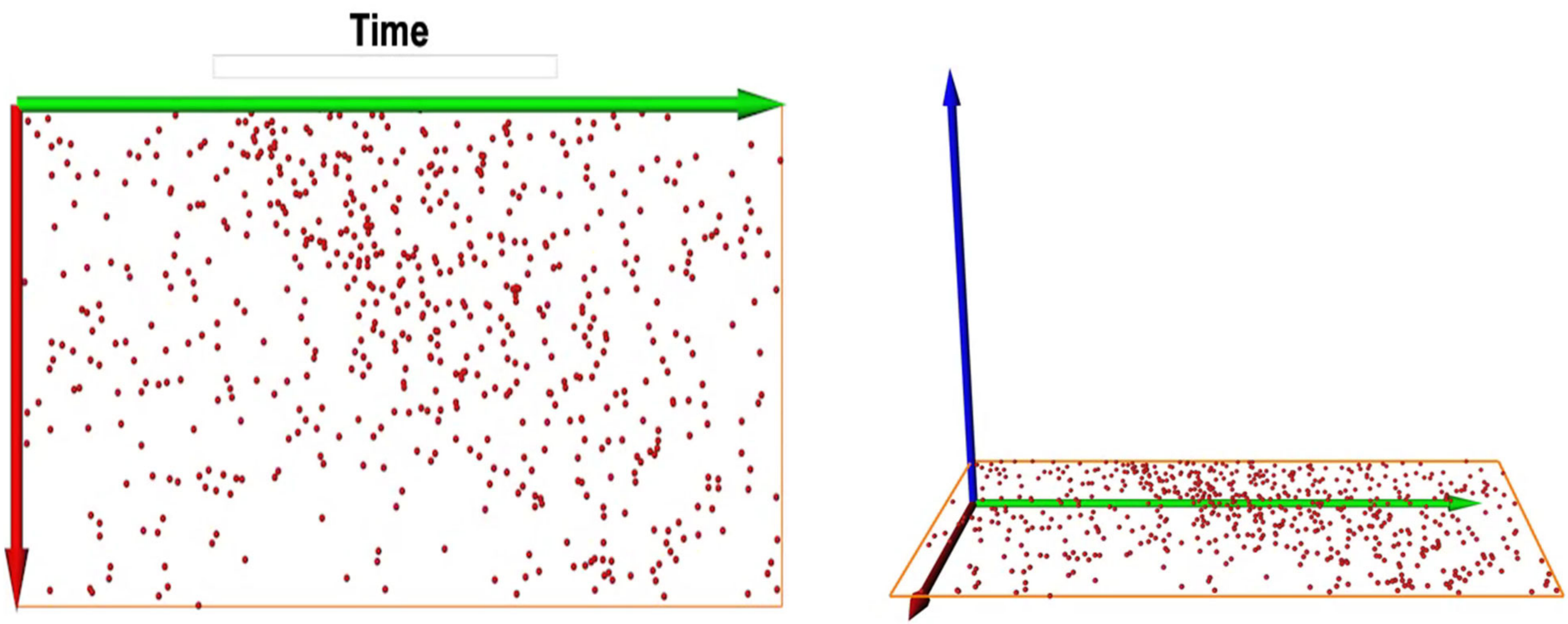
(2) Streamlines:
- are only geometrically, not physically meaningful
- are easiest to compute (no temporal interpolation, single IVP)
- are better suited for static visualization (prints)
- don't intersect (under reasonable assumptions)

(3) Streaklines:

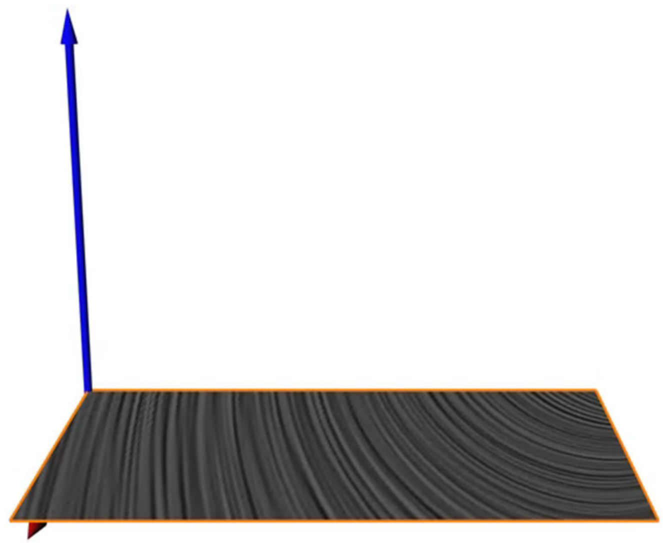
- are physically meaningful
- allow comparison with experiment (dye injection)
- are well suited for static and dynamic visualization
- good choice for fast moving vortices
- can be approximated by set of disconnected particles

(4) Timelines:

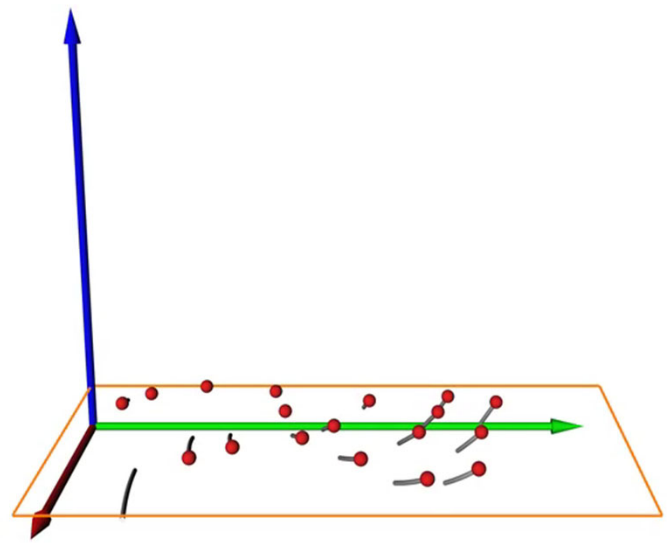
- are physically meaningful
- are well suited for static and dynamic visualization
- can be approximated by set of disconnected particles



2D time-dependent vector field
particle visualization



stream lines



path lines

curve parallel to the vector field in each point for a **fixed time**

describes motion of a massless particle in an **steady** flow field

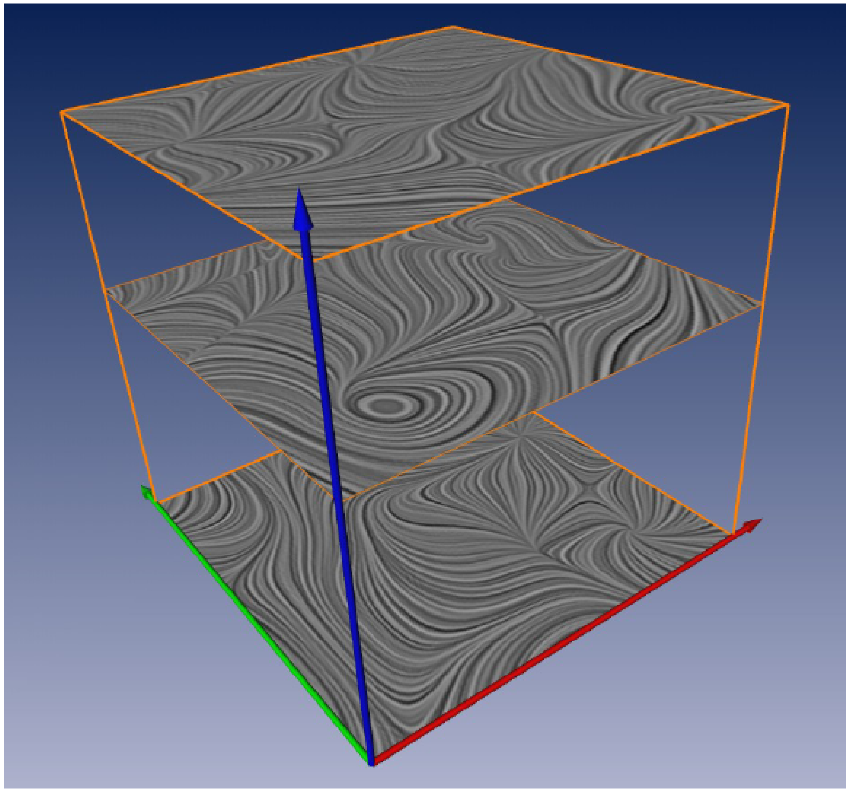
curve parallel to the vector field in each point **over time**

describes motion of a massless particle in an **unsteady** flow field

Streamlines Over Time

Defined only for steady flow or for a fixed time step (of unsteady flow)

Different tangent curves in every time step for time-dependent vector fields (unsteady flow)

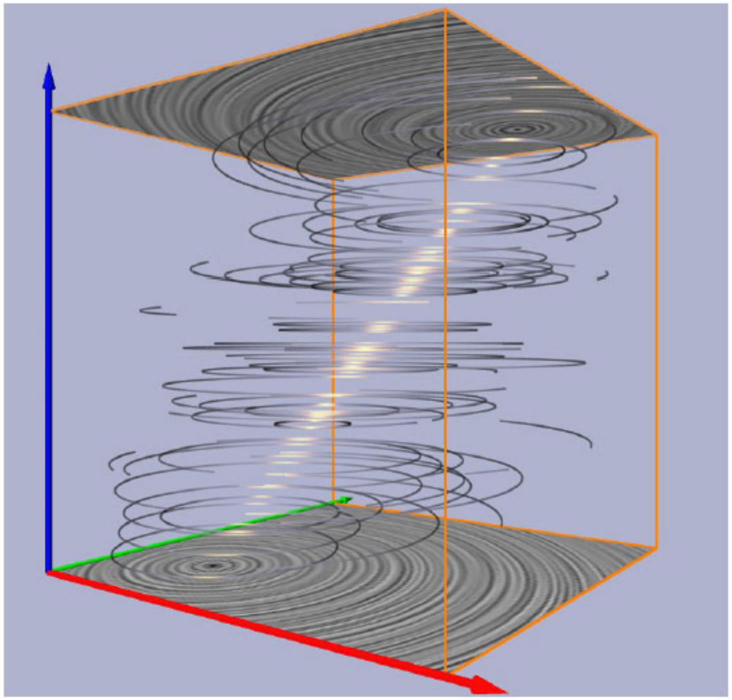


Tino Weinkauf

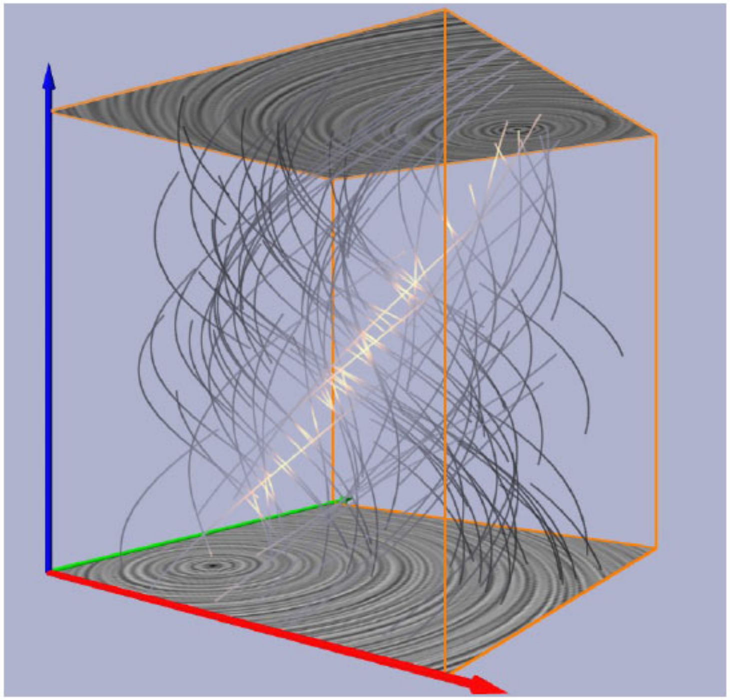
Stream Lines vs. Path Lines Viewed Over Time

Plotted with time as third dimension

- Tangent curves to a (n + 1)-dimensional vector field



Stream Lines



Path Lines

Vector fields as ODEs

For simplicity, the vector field is now interpreted as a velocity field.

Then the field $\mathbf{v}(\mathbf{x},t)$ describes the connection between location and velocity of a (massless) particle.

It can equivalently be expressed as an ordinary differential equation

$$\dot{\mathbf{x}}(t)=\mathbf{v}(\mathbf{x}(t),t)$$

This ODE, together with an initial condition

$$\mathbf{x}(t_0)=\mathbf{x}_0 \ ,$$

is a so-called initial value problem (IVP).

Its solution is the integral curve (or trajectory)

$$\mathbf{x}(t)=\mathbf{x}_0+\int_{t_0}^{t}\mathbf{v}(\mathbf{x}(\tau),\tau)\,d\tau$$

The integral curve is a pathline, describing the path of a massless particle which was released at time t_0 at position x_0.

Remark: $t < t_0$ is allowed.

For static fields, the ODE is autonomous:

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t))$$

and its integral curves

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(\tau))\,d\tau$$

are called field lines, or (in the case of velocity fields) streamlines.

In static vector fields, pathlines and streamlines are identical.

In time-dependent vector fields, instantaneous streamlines can be computed from a "snapshot" at a fixed time T (which is a static vector field)

$$\mathbf{v}_T(\mathbf{x}) = \mathbf{v}(\mathbf{x}, T)$$

In practice, time-dependent fields are often given as a dataset per time step. Each dataset is then a snapshot.

Streamline integration

Outline of algorithm for numerical streamline integration (with obvious extension to pathlines):

Inputs:

- static vector field $\mathbf{v}(\mathbf{x})$
- seed points with time of release $(\mathbf{x}_0, t_0)$
- control parameters:
  - step size (temporal, spatial, or in local coordinates)
  - step count limit, time limit, etc.
  - order of integration scheme

Output:

- streamlines as "polylines", with possible attributes (interpolated field values, time, speed, arc length, etc.)

Preprocessing:

- set up search structure for point location
- for each seed point:
  - global point location: Given a point **x**,
    find the cell containing **x** and the local coordinates (ξ, η, ζ)
    or ir the grid is structured:
    find the computational space coordinates $(i + \xi,\ j + \eta,\ k + \zeta)$
  - If **x** is not found in a cell, remove seed point

Integration loop, for each seed point **x**:

- interpolate ***v*** trilinearly to local coordinates (ξ, η, ζ)
- do an integration step, producing a new point **x'**
- incremental point location: For position **x'** find cell and local coordinates (ξ', η', ζ') making use of information (coordinates, local coordinates, cell) of old point **x**

Termination criteria:

- grid boundary reached
- step count limit reached
- optional: velocity close to zero
- optional: time limit reached
- optional: arc length limit reached

Integration step: widely used integration methods:

- Euler (used only in special speed-optimized techniques, e.g. GPU-based texture advection)

$$\mathbf{x}_{new} = \mathbf{x} + \mathbf{v}(\mathbf{x}, t) \cdot \Delta t$$

- Runge-Kutta, 2nd or 4th order

Higher order than 4th?

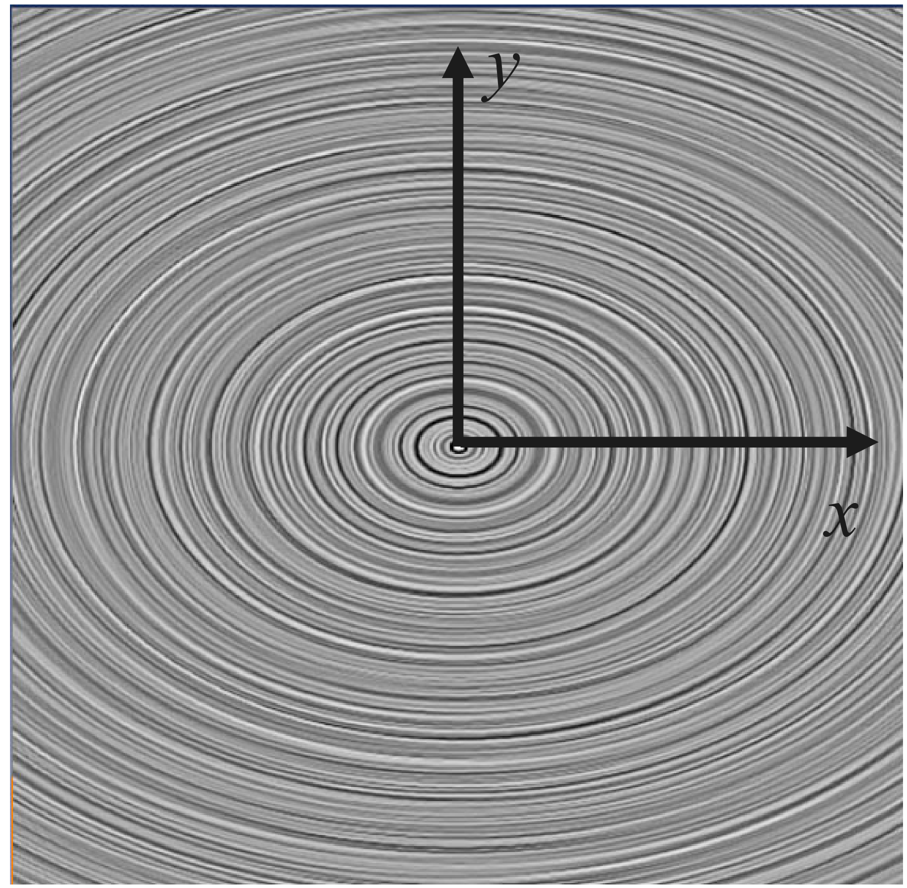
- often too slow for visualization
- study (Yeung/Pope 1987) shows that, when using standard trilinear interpolation, interpolation errors dominate integration errors.

Numerical Integration

- **Numerical integration of stream lines:**

- approximate streamline by polygon $\mathbf{x}_i$

- **Testing example:**
  - $\mathbf{v}(x,y) = (-y, x/2)$^$T$
  - exact solution: ellipses
  - starting integration from (0,-1)

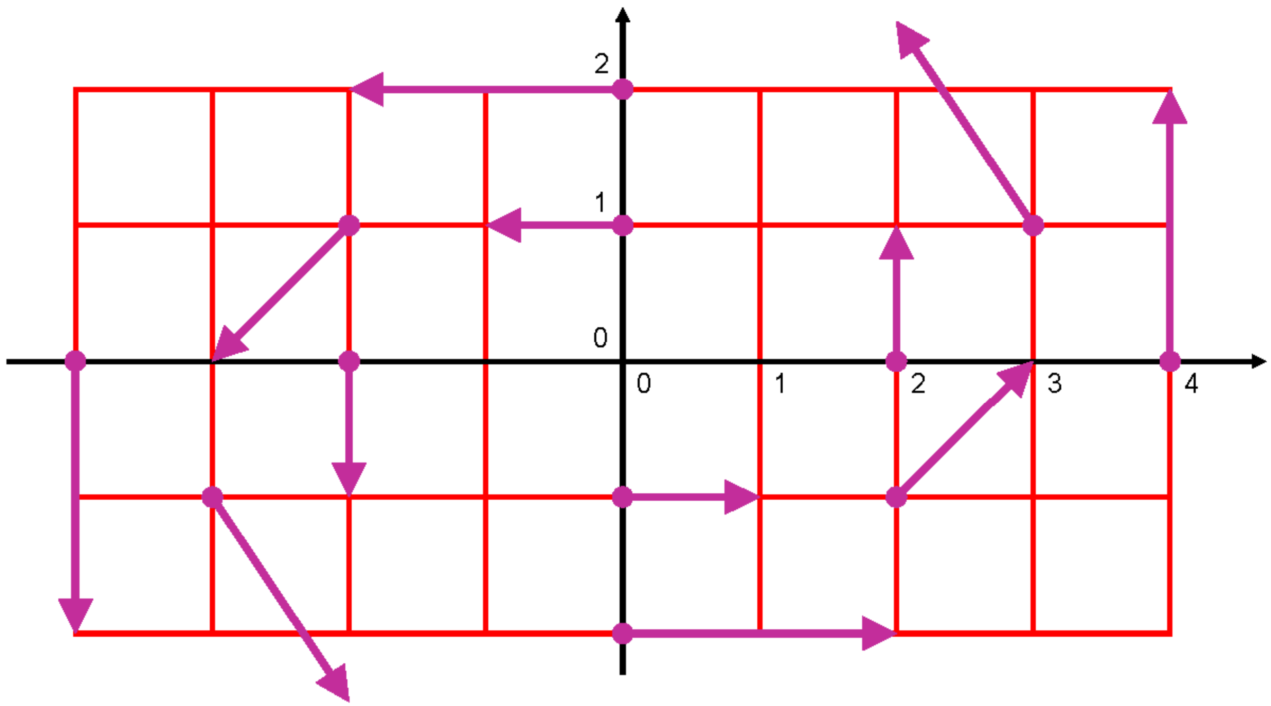


Streamlines – Practice

- Basic approach:
  - theory: $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{s}(u))\, \mathrm{d}u$
  - practice: numerical integration
  - idea:
    (very) locally, the solution is (approx.) linear
  - Euler integration:
    follow the current flow vector $\mathbf{v}(\mathbf{s}_i)$ from the current streamline point $\mathbf{s}_i$ for a very small time ($\mathrm{d}t$) and therefore distance
  - Euler integration: $\mathbf{s}_{i+1} = \mathbf{s}_i + \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i)$,
    integration of small steps ($\mathrm{d}t$ very small)

Euler Integration – Example

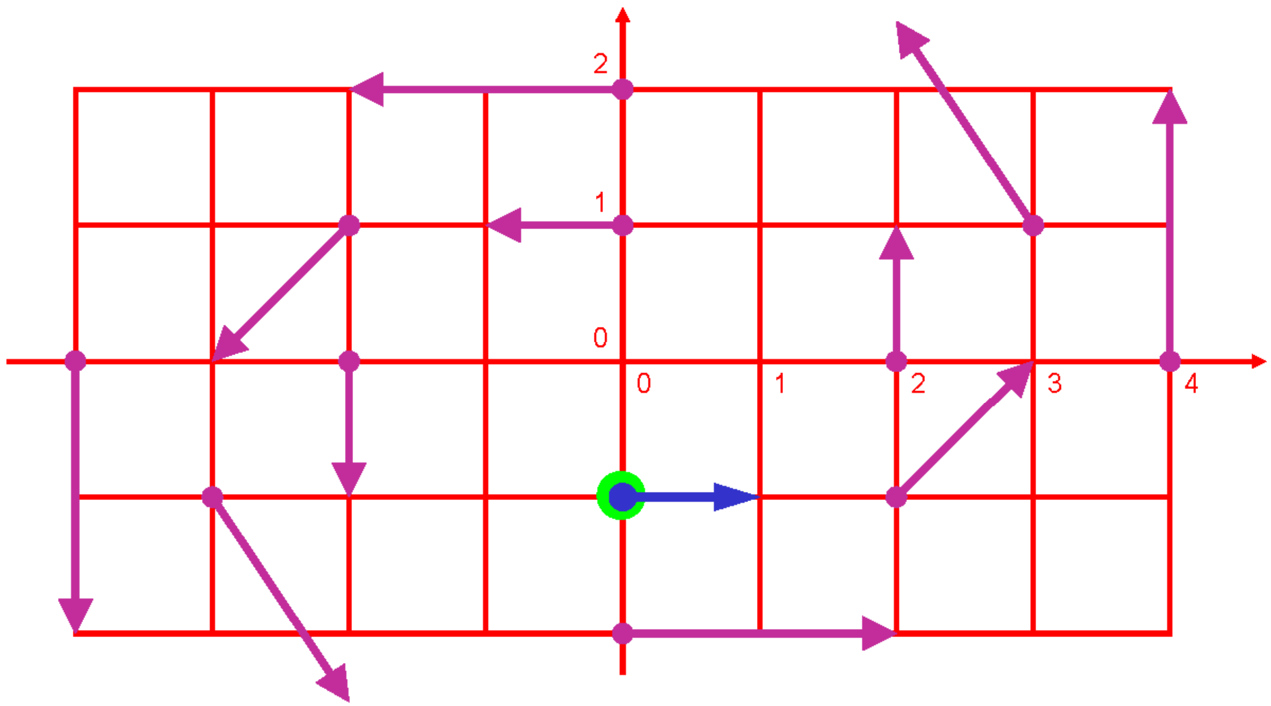
- 2D model data: $v_x = dx/dt = -y$
  $v_y = dy/dt = x/2$
- Sample arrows:



- True solution: ellipses!

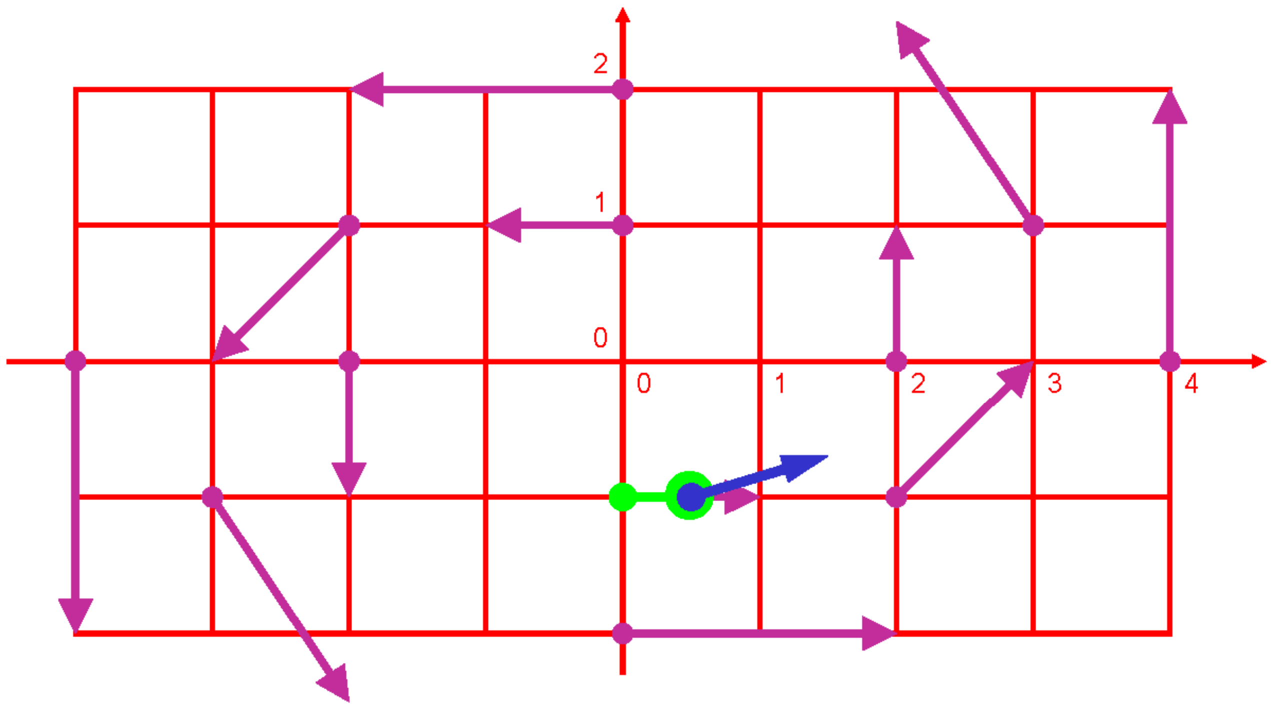
Euler Integration – Example

- Seed point $\mathbf{s}_0 = (0\,|\,-1)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_0) = (1\,|\,0)^T$;
  $dt = 1/2$



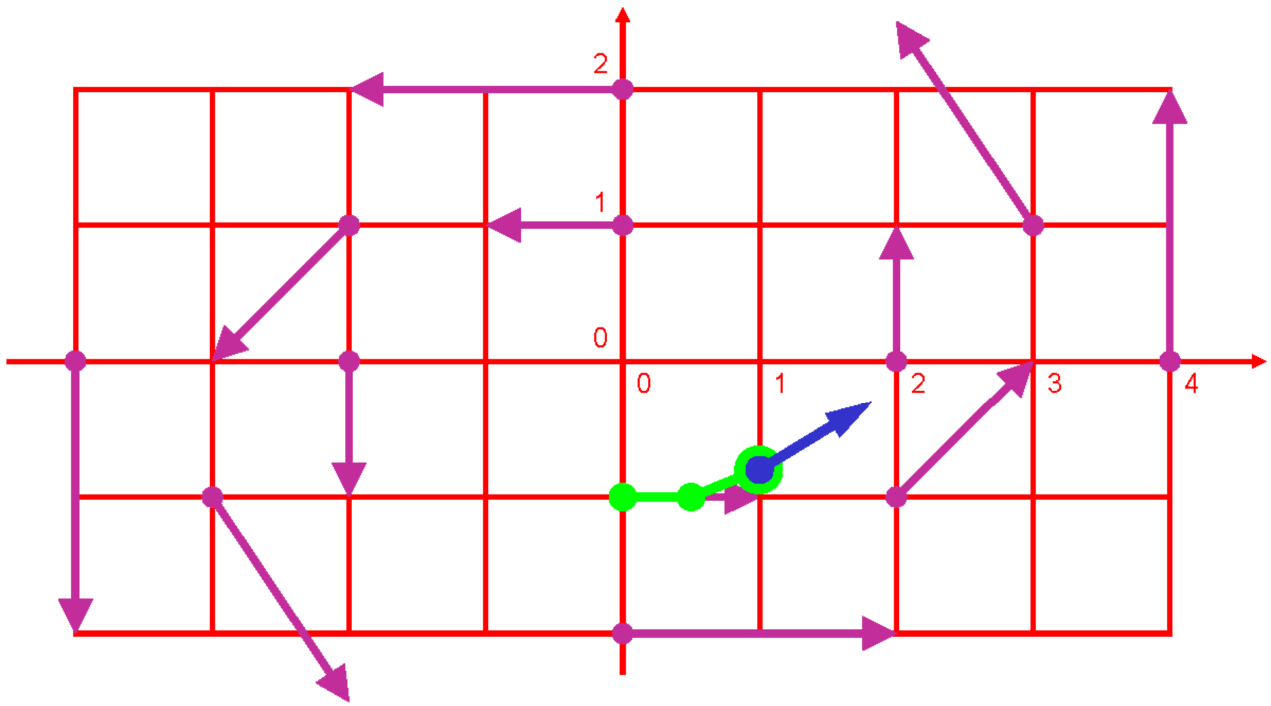
Euler Integration – Example

- New point $\mathbf{s}_1 = \mathbf{s}_0 + \mathbf{v}(\mathbf{s}_0) \cdot \mathrm{d}t = (\,1/2\,|\,-1\,)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_1) = (\,1\,|\,1/4\,)^T$;



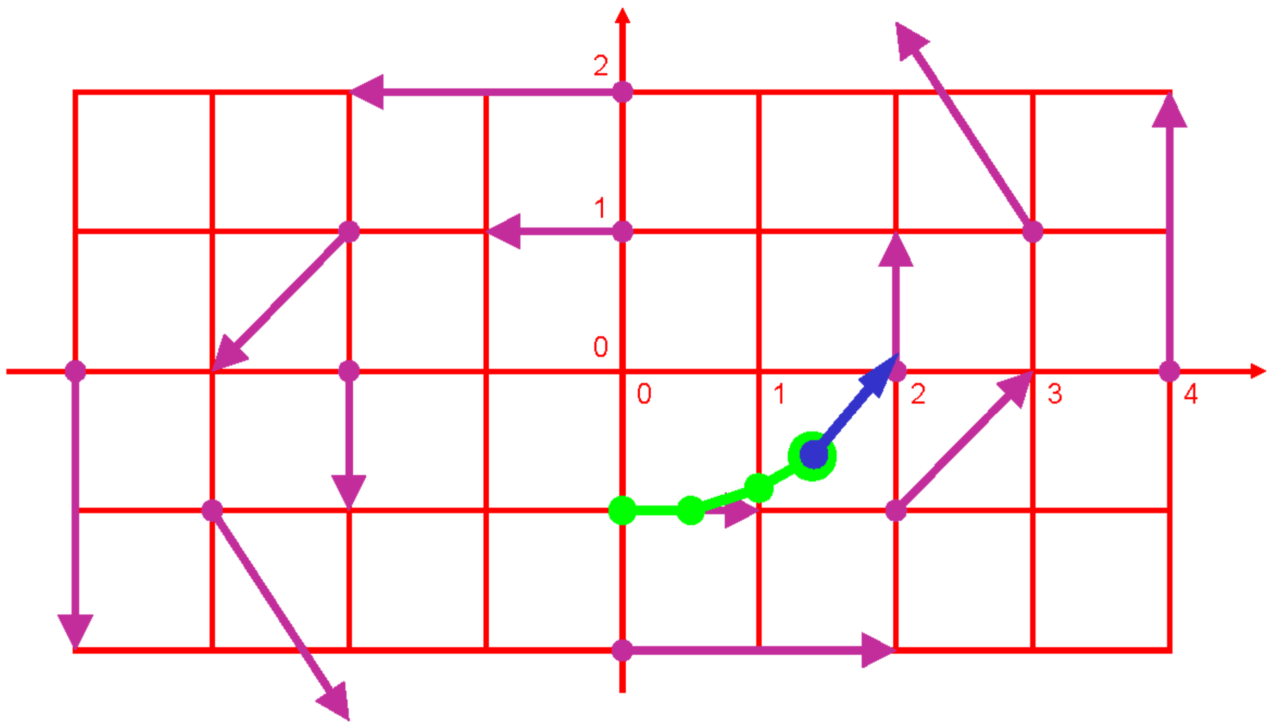
Euler Integration – Example

- New point $\mathbf{s}_2 = \mathbf{s}_1 + \mathbf{v}(\mathbf{s}_1) \cdot dt = (1\,|\,-7/8)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_2) = (7/8\,|\,1/2)^T$;



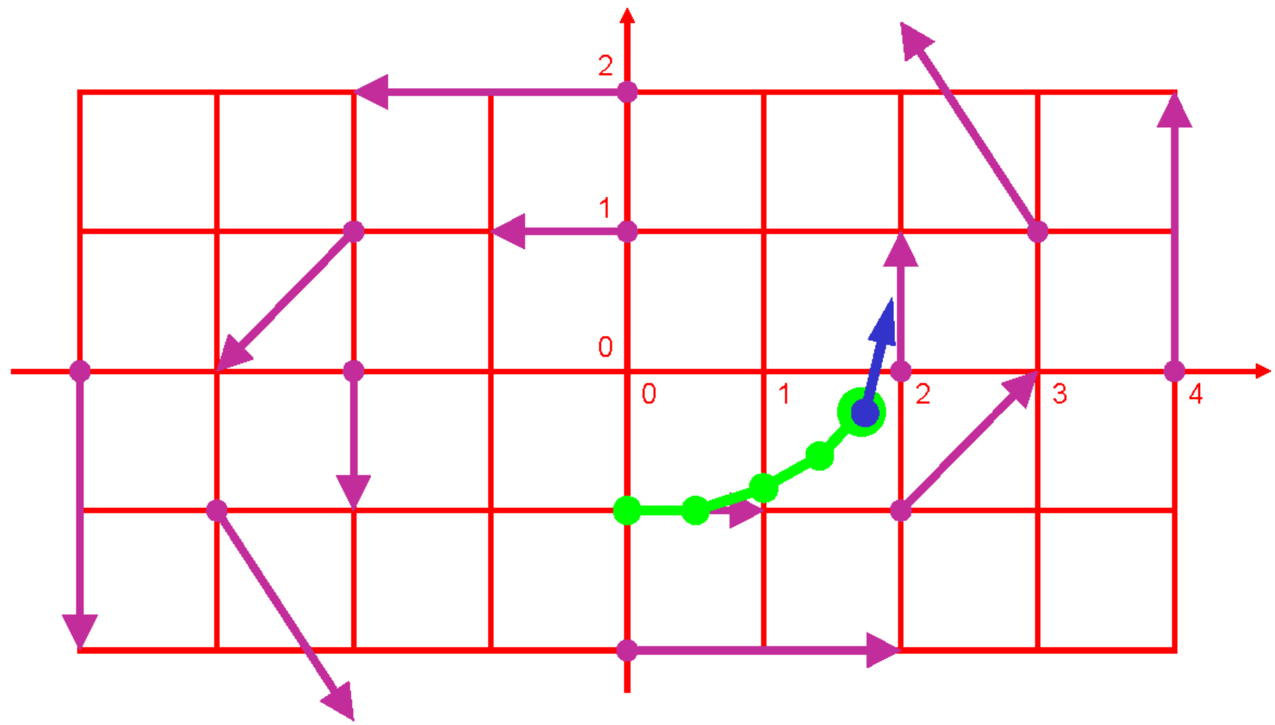
Euler Integration – Example

- $\mathbf{s}_3 = (23/16 \mid -5/8)^T \approx (1.44 \mid -0.63)^T;$
  $\mathbf{v}(\mathbf{s}_3) = (5/8 \mid 23/32)^T \approx (0.63 \mid 0.72)^T;$



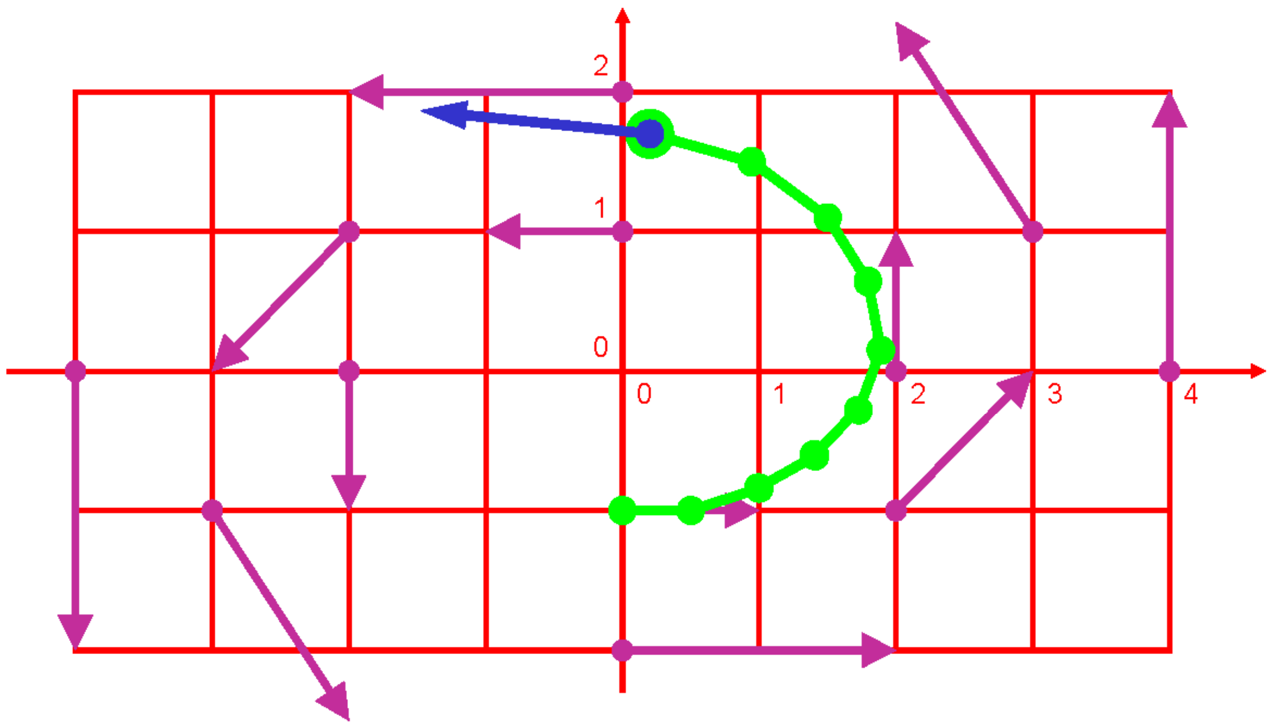
Euler Integration – Example

- $\mathbf{s}_4 = (7/4 \,|\, -17/64)^T \approx (1.75 \,|\, -0.27)^T;$
  $\mathbf{v}(\mathbf{s}_4) = (17/64 \,|\, 7/8)^T \approx (0.27 \,|\, 0.88)^T;$



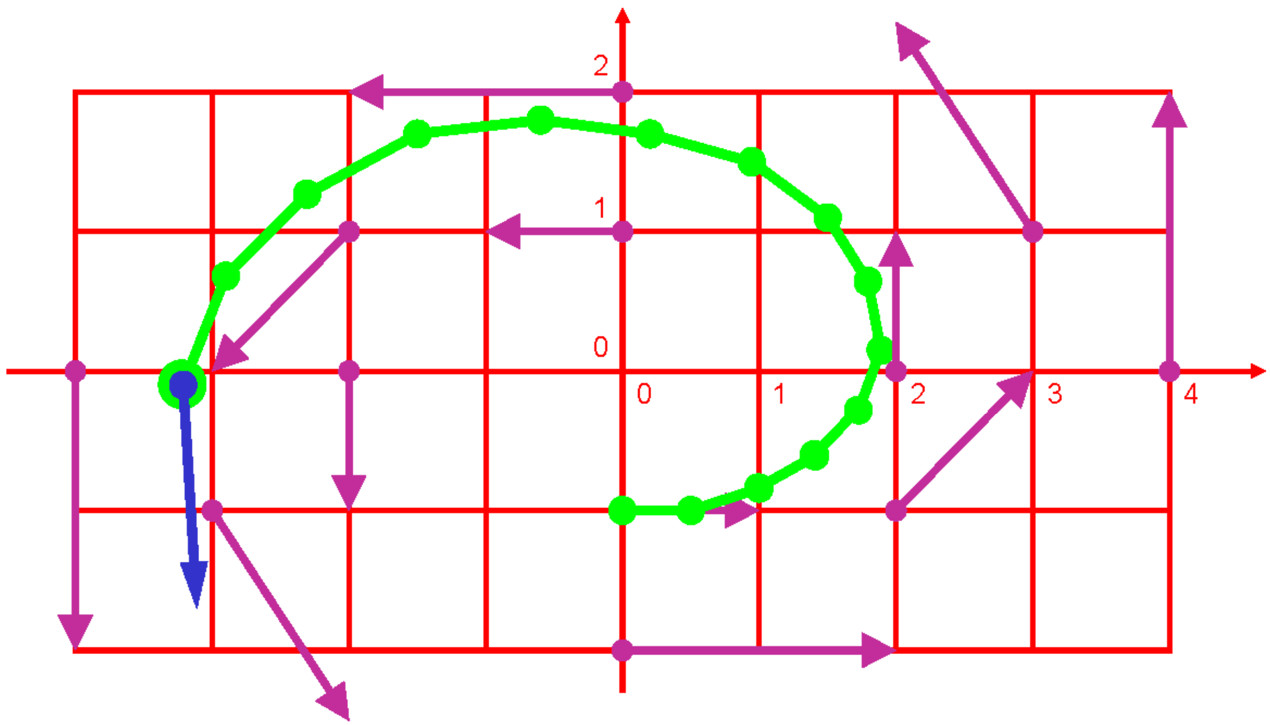
Euler Integration – Example

- $\mathbf{s}_9 \approx (0.20\,|\,1.69)^T$;
  $\mathbf{v}(\mathbf{s}_9) \approx (-1.69\,|\,0.10)^T$;



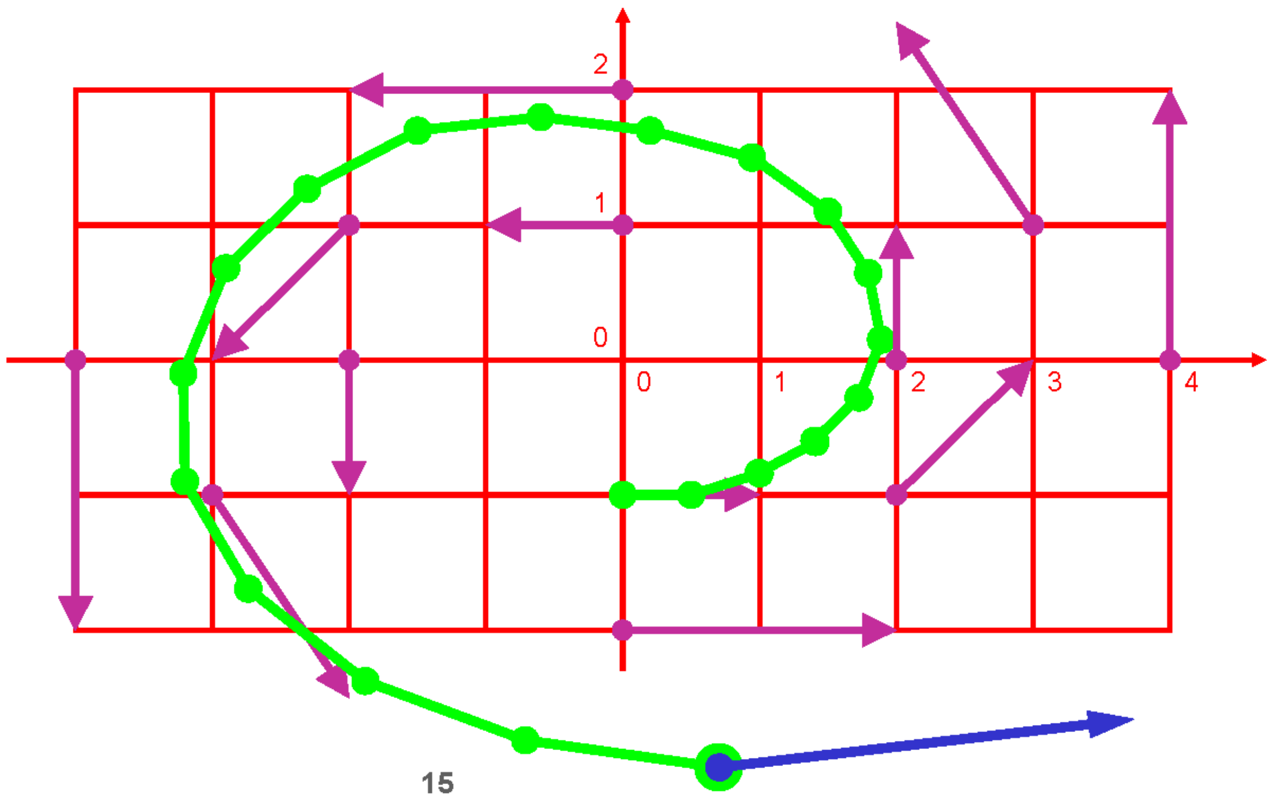
Euler Integration – Example

- $\mathbf{s}_{14} \approx (-3.22|-0.10)^T$;
  $\mathbf{v}(\mathbf{s}_{14}) \approx (0.10|-1.61)^T$;



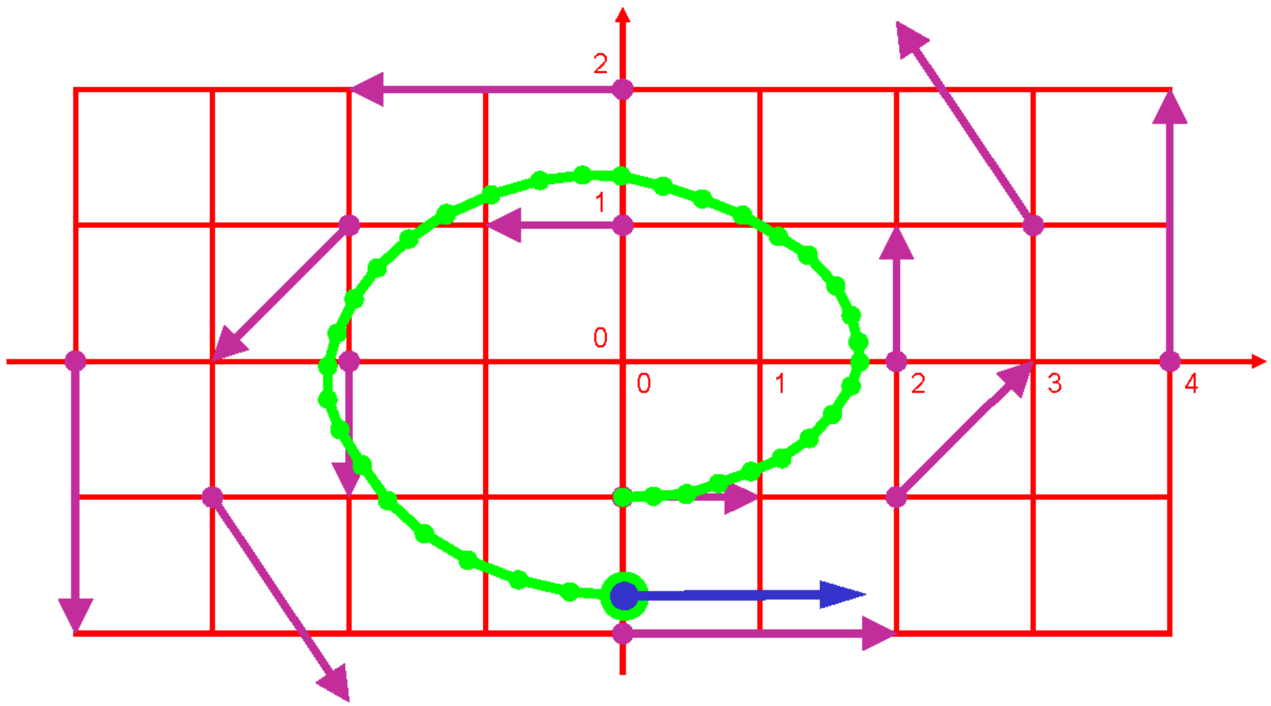
Euler Integration – Example

- $\mathbf{s}_{19} \approx (0.75|-3.02)^T$; $\mathbf{v}(\mathbf{s}_{19}) \approx (3.02|0.37)^T$;
  clearly: large integration error, dt too large!
  19 steps



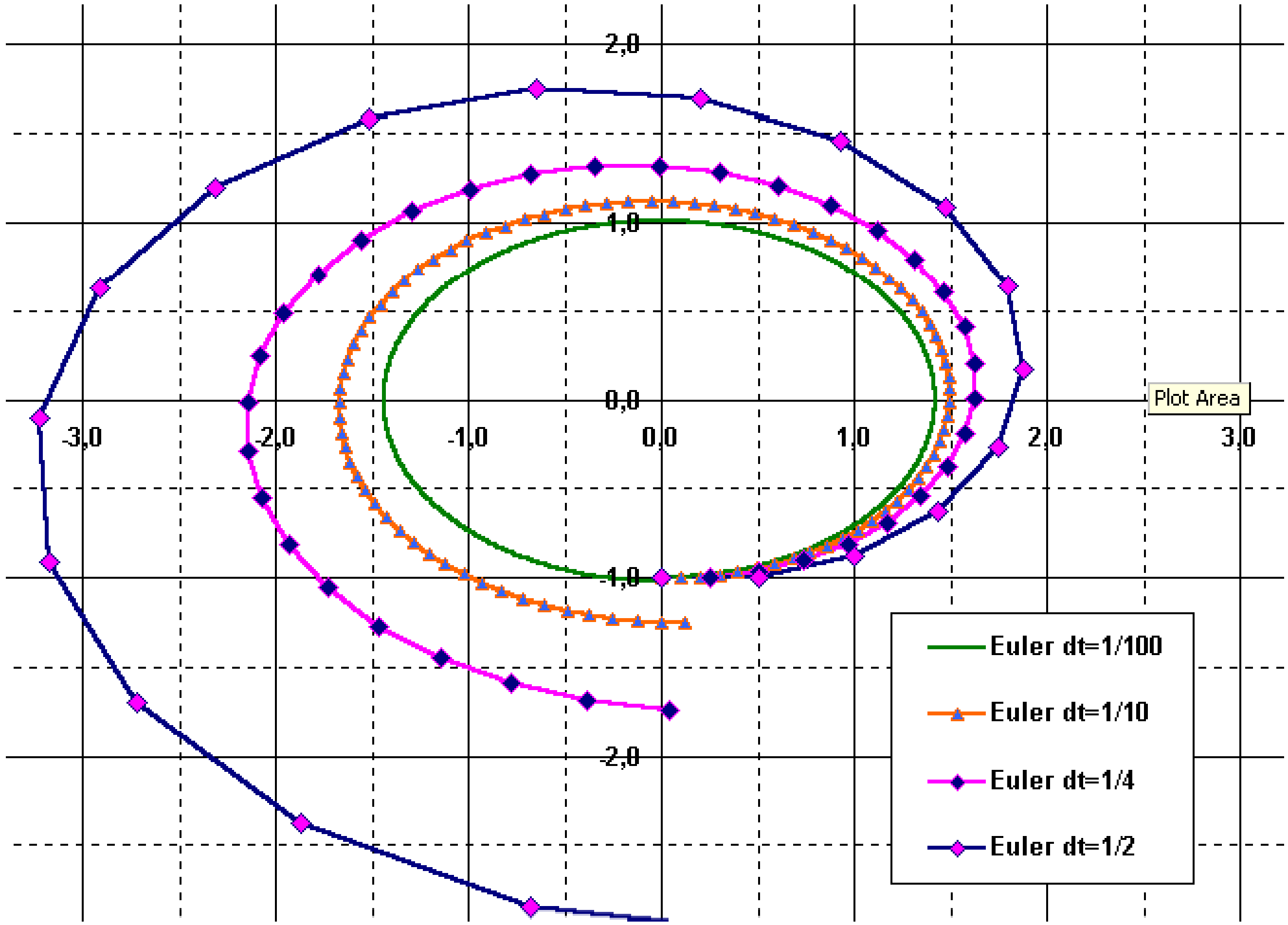
Euler Integration – Example

- d*t* smaller (1/4): more steps, more exact!
  $\mathbf{s}_{36} \approx (0.04 \,|\, -1.74)^T$; $\mathbf{v}(\mathbf{s}_{36}) \approx (1.74 \,|\, 0.02)^T$;
- 36 steps



Comparison Euler, Step Sizes

Euler
is getting
better
propor-
tionally
to d*t*

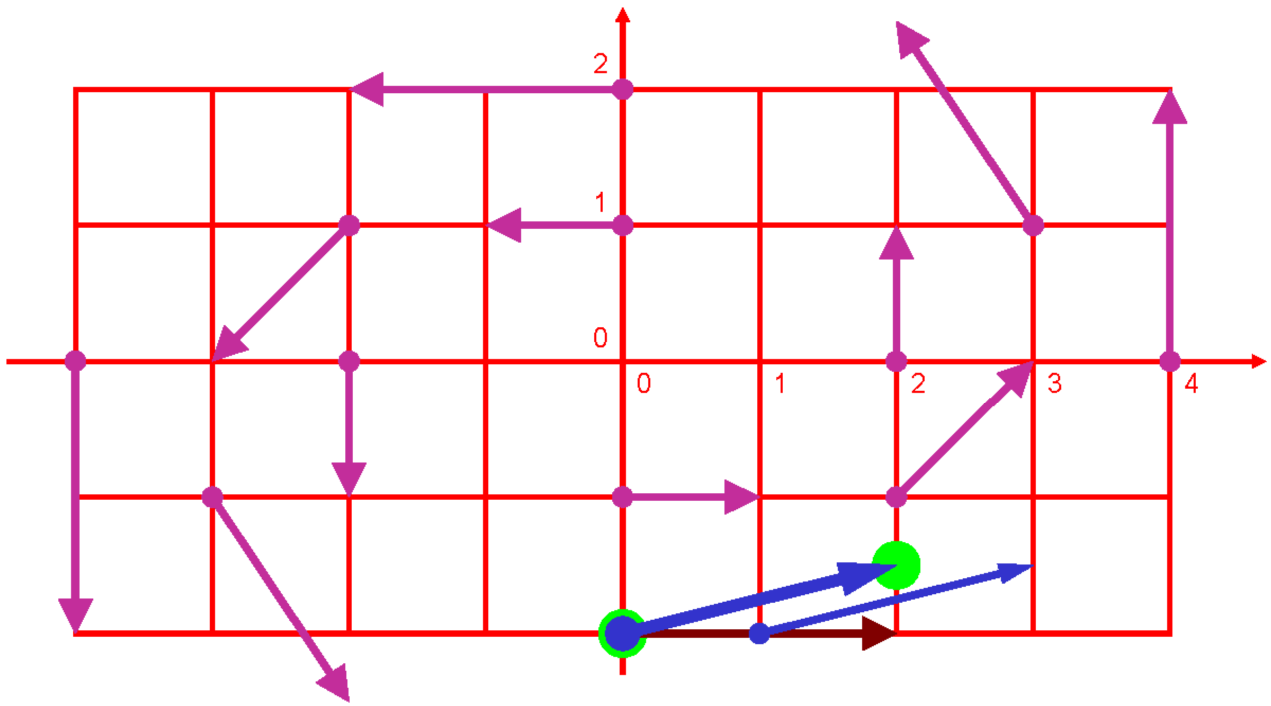


Better than Euler Integr.: RK

- Runge-Kutta Approach:
  - theory: $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{s}(u))\, \mathrm{d}u$
  - Euler: $\mathbf{s}_i = \mathbf{s}_0 + \Sigma_{0 \le u < i} \mathbf{v}(\mathbf{s}_u) \cdot \mathrm{d}t$
  - Runge-Kutta integration:
    - idea: cut short the curve arc
    - RK-2 (second order RK):
      1.: do half a Euler step
      2.: evaluate flow vector there
      3.: use it in the origin
    - RK-2 (two evaluations of **v** per step):
      $\mathbf{s}_{i+1} = \mathbf{s}_i + \mathbf{v}(\mathbf{s}_i + \mathbf{v}(\mathbf{s}_i) \cdot \mathrm{d}t/2) \cdot \mathrm{d}t$

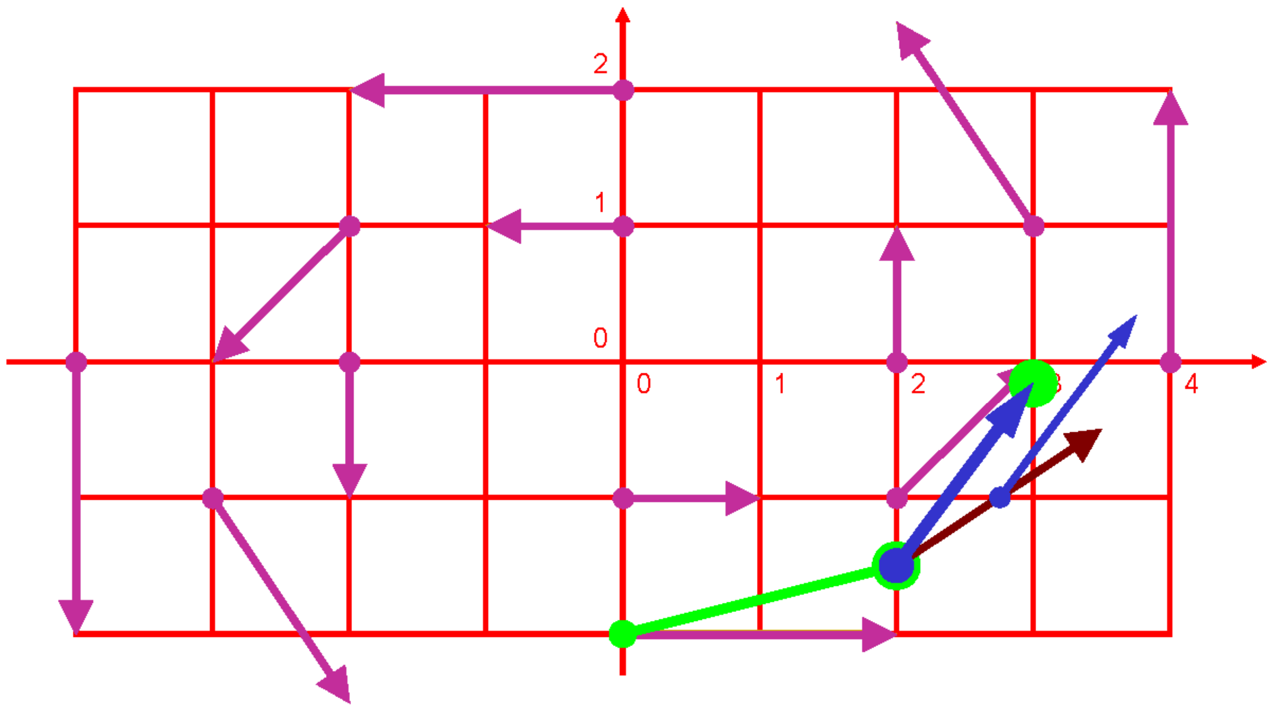
RK-2 Integration – One Step

- Seed point $\mathbf{s}_0 = (0|-2)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_0) = (2|0)^T$;
  preview vector $\mathbf{v}(\mathbf{s}_0+\mathbf{v}(\mathbf{s}_0)\cdot dt/2) = (2|0.5)^T$;
  $dt = 1$



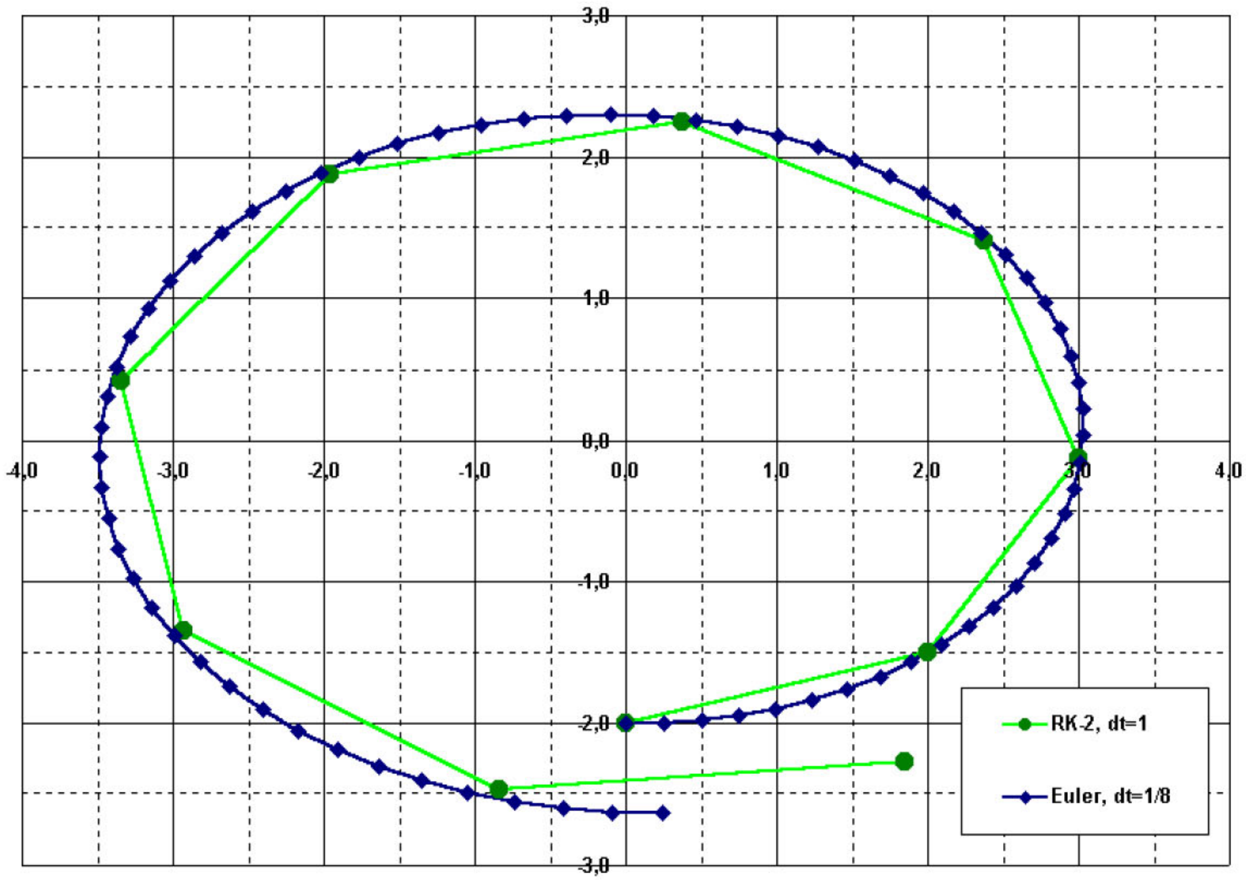
RK-2 – One more step

- Seed point $\mathbf{s}_1 = (2|-1.5)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_1) = (1.5|1)^T$;
  preview vector $\mathbf{v}(\mathbf{s}_1+\mathbf{v}(\mathbf{s}_1)\cdot dt/2) \approx (1|1.4)^T$;
  $dt = 1$



RK-2 – A Quick Round

- RK-2: even with dt=1 (9 steps) better than Euler with dt=1/8 (72 steps)

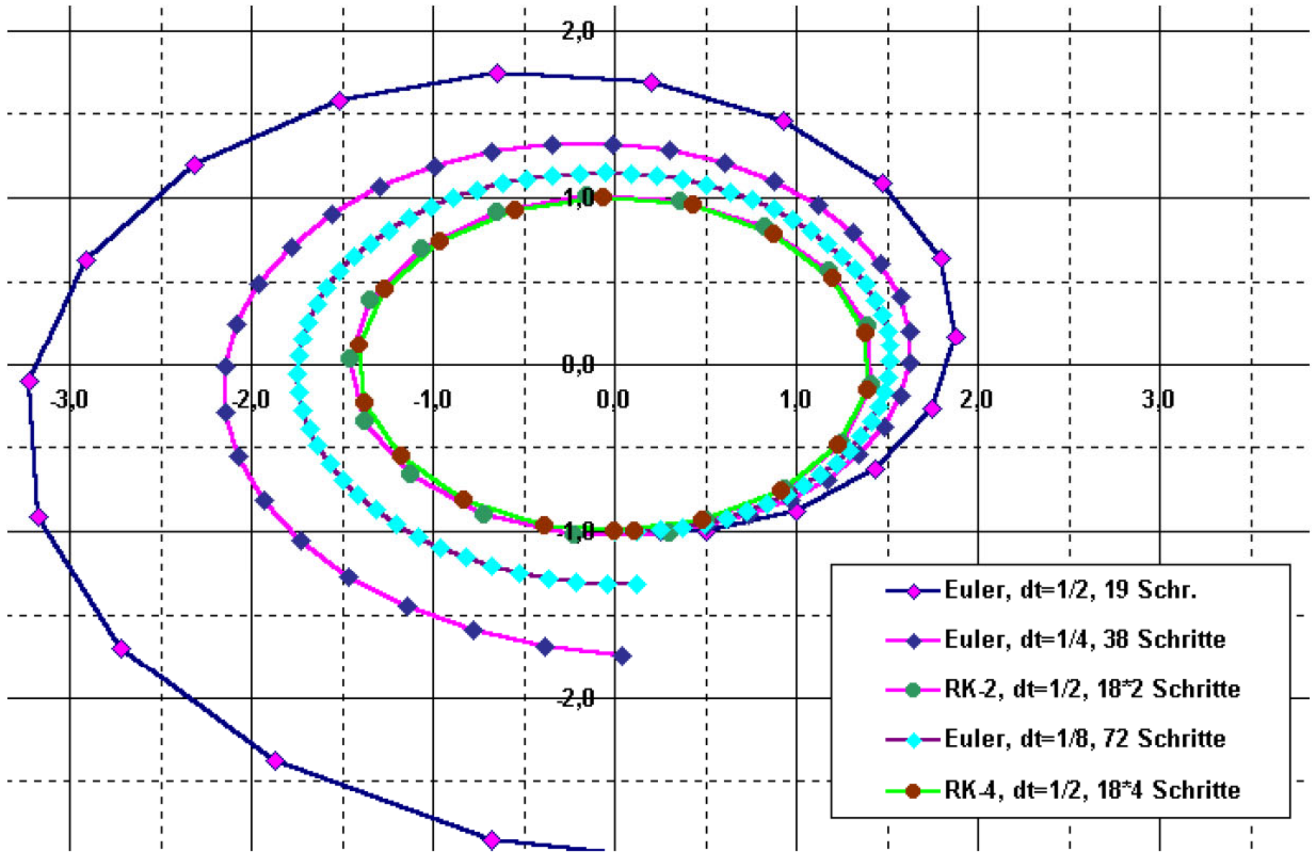


RK-4 vs. Euler, RK-2

- Even better: fourth order RK:
  - four vectors **a**, **b**, **c**, **d**
  - one step is a convex combination:
    $\mathbf{s}_{i+1} = \mathbf{s}_i + (\mathbf{a} + 2 \cdot \mathbf{b} + 2 \cdot \mathbf{c} + \mathbf{d})/6$
  - vectors:
    - $\mathbf{a} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i)$ … original vector
    - $\mathbf{b} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{a}/2)$ … RK-2 vector
    - $\mathbf{c} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{b}/2)$ … use RK-2 …
    - $\mathbf{d} = \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{c})$ … and again!

Euler vs. Runge-Kutta

- RK-4: pays off only with complex flows
- Here approx. like RK-2



Integration, Conclusions

- Summary:
  - analytic determination of streamlines usually not possible
  - hence: numerical integration
  - several methods available (Euler, Runge-Kutta, etc.)
  - Euler: simple, imprecise, esp. with small dt
  - RK: more accurate in higher orders
  - furthermore: adaptive methods, implicit methods, etc.

Thank you.

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