King Abdullah University of Science and Technology KAUST

CS 247 – Scientific Visualization
Lecture 10: Scalar Field Visualization, Pt. 4

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Reading Assignment #5 (until Mar 5)

Read (required):

- Gradients of scalar-valued functions

  https://en.wikipedia.org/wiki/Gradient

- Critical points

  https://en.wikipedia.org/wiki/Critical_point_(mathematics)

- Multivariable derivatives and differentials

  https://en.wikipedia.org/wiki/Total_derivative

  https://en.wikipedia.org/wiki/Differential_of_a_function#Differentials_in_several_variables

  https://en.wikipedia.org/wiki/Hessian_matrix

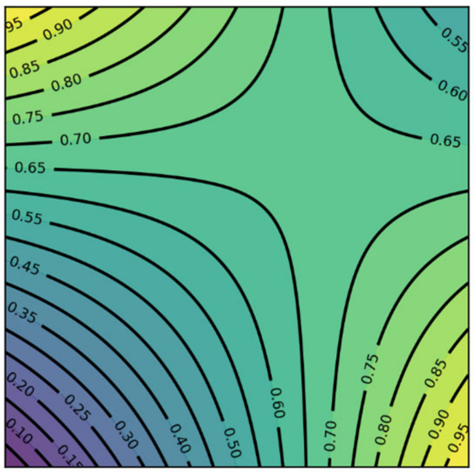
- Dot product, inner product (more general)

  https://en.wikipedia.org/wiki/Dot_product

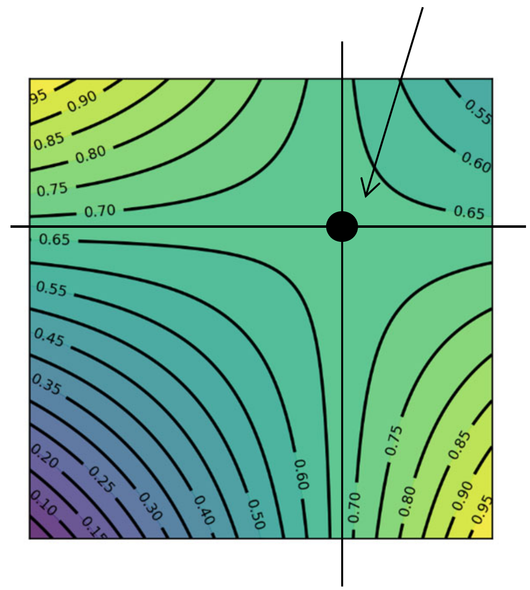
  https://en.wikipedia.org/wiki/Inner_product_space

Bi-Linear Interpolation: Critical Points

Critical points are where the gradient vanishes (i.e., is the zero vector)



critical point
(saddle point)



here, the critical
value is 2/3=0.666…

“Asymptotic decider”: resolve ambiguous configurations (6 and 9) by
comparing specific iso-value with critical value (scalar value at critical point)

Bi-Linear Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers):

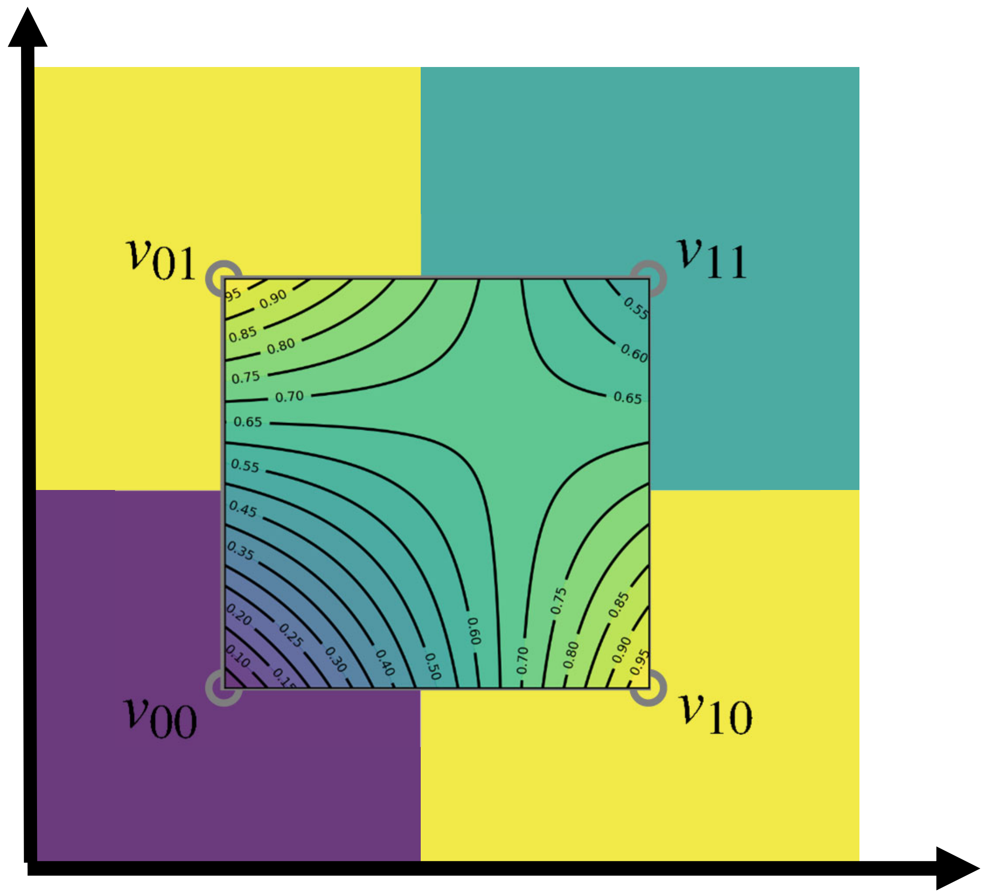
Given any (fractional) position

$$\alpha_1 := x_1 - \lfloor x_1 \rfloor \qquad \alpha_1 \in [0.0, 1.0)$$
$$\alpha_2 := x_2 - \lfloor x_2 \rfloor \qquad \alpha_2 \in [0.0, 1.0)$$

and 2x2 sample values

$$\begin{bmatrix} v_{01} & v_{11} \\ v_{00} & v_{10} \end{bmatrix}$$

Compute: $f(\alpha_1, \alpha_2)$



Bi-Linear Interpolation

Interpolate function at (fractional) position (α_1, α_2) :

$$f(\alpha_1, \alpha_2) = \begin{bmatrix} \alpha_2 & (1-\alpha_2) \end{bmatrix} \begin{bmatrix} v_{01} & v_{11} \\ v_{00} & v_{10} \end{bmatrix} \begin{bmatrix} (1-\alpha_1) \\ \alpha_1 \end{bmatrix}$$

$$= (1-\alpha_1)(1-\alpha_2)v_{00} + \alpha_1(1-\alpha_2)v_{10} + (1-\alpha_1)\alpha_2 v_{01} + \alpha_1\alpha_2 v_{11}$$

$$= v_{00} + \alpha_1(v_{10} - v_{00}) + \alpha_2(v_{01} - v_{00}) + \alpha_1\alpha_2(v_{00} + v_{11} - v_{10} - v_{01})$$

Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2 (v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1 (v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0: \qquad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

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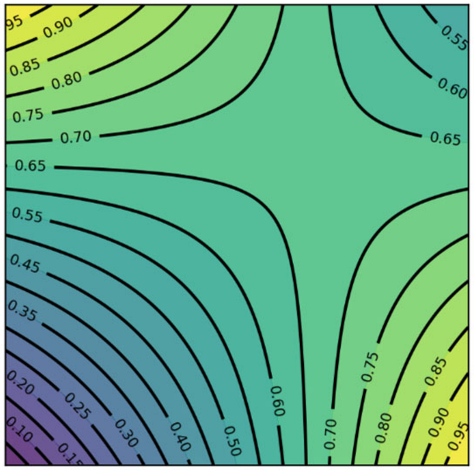


if denominator is zero, bi-linear interpolation has degenerated to linear interpolation (or const)! (also means: no isolated critical points!)

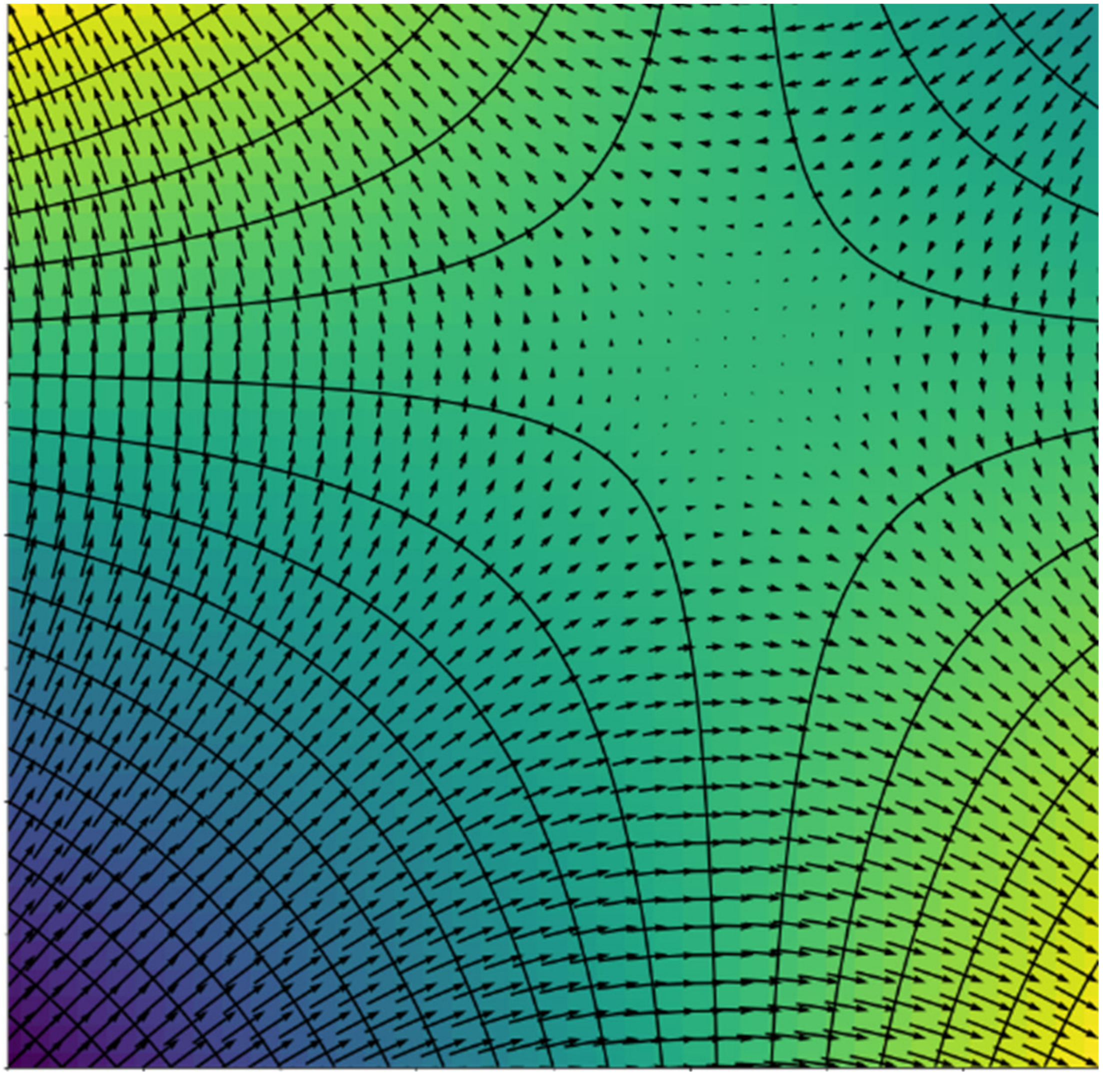
Bi-Linear Interpolation: Critical Points

Compute gradient

Note that isolines are farther apart where gradient is smaller

Note the horizontal and vertical lines where gradient becomes vertical/horizontal

Note the critical point



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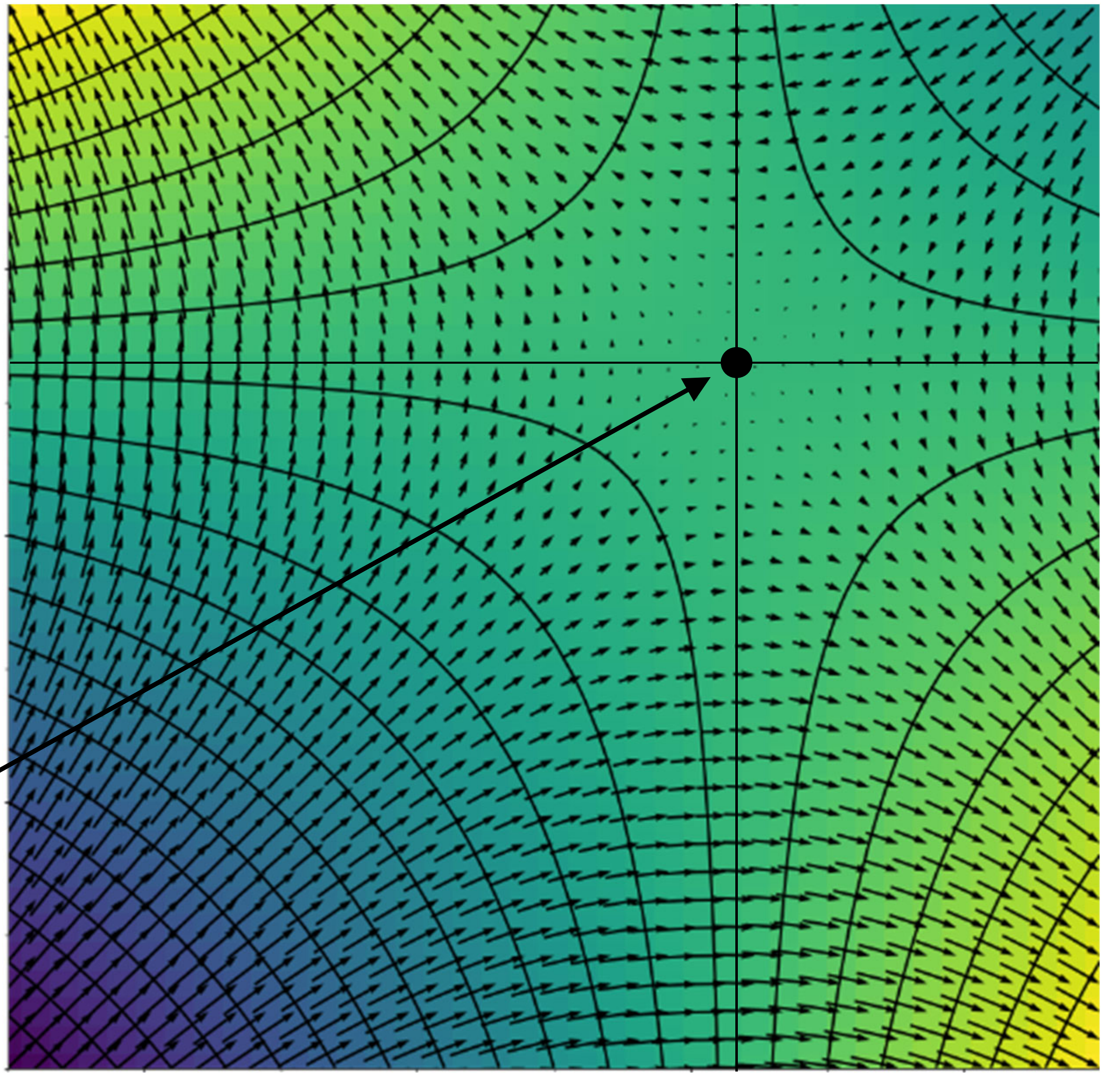

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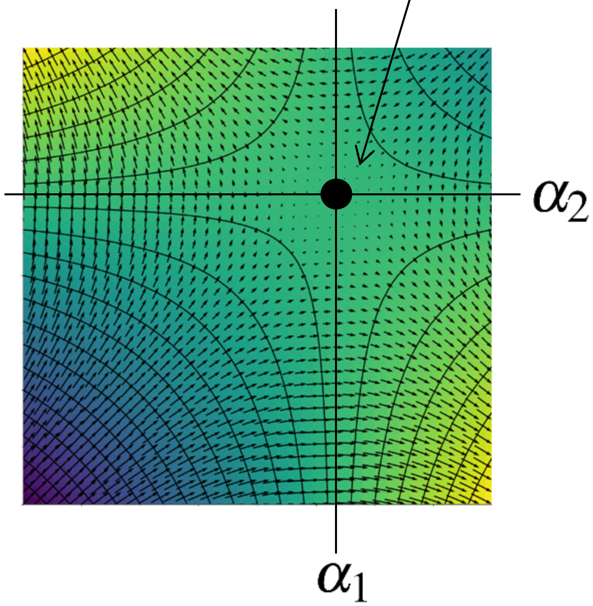

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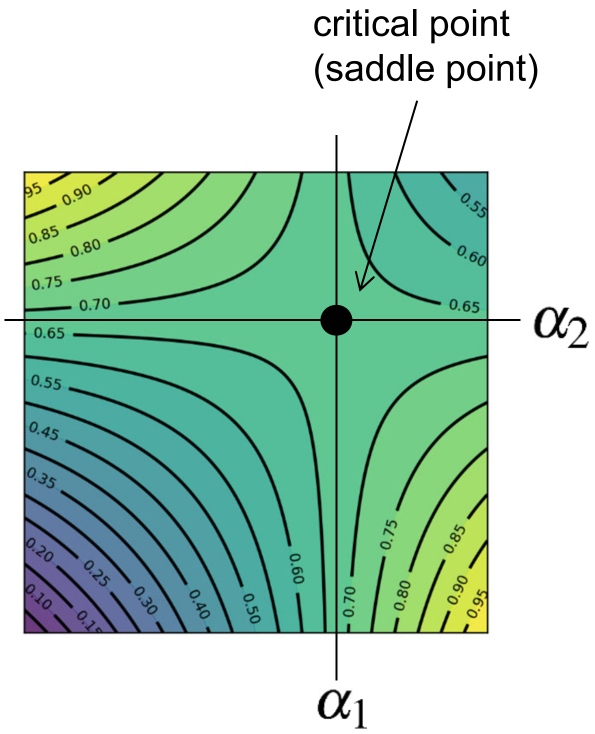


Bi-Linear Interpolation: Critical Points

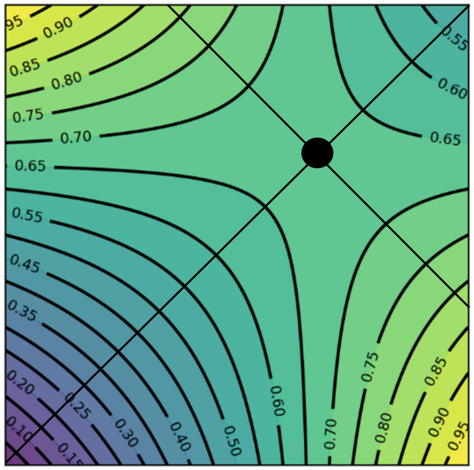
Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

$$\begin{bmatrix} \frac{\partial^2 f}{\partial \alpha_1^2} & \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \qquad a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



(here also: principal curvature magnitudes and directions
of this function's graph == surface embedded in 3D)

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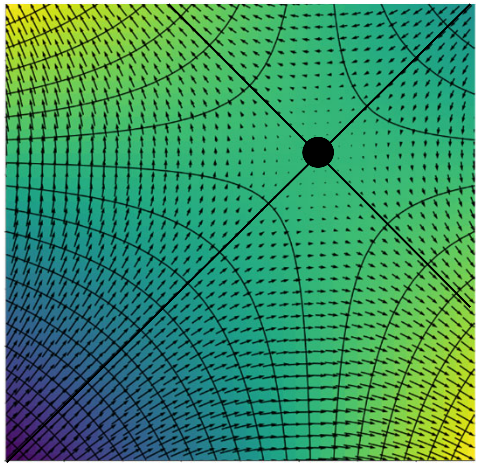

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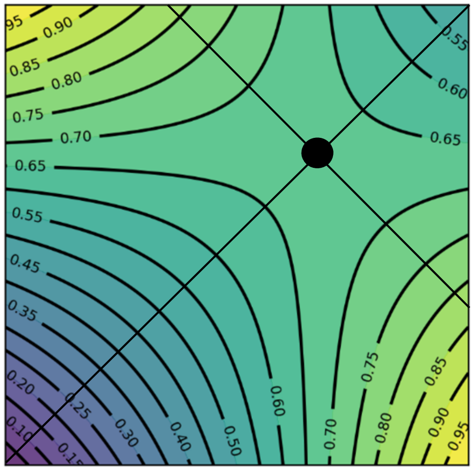
$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

degenerate means determinant = 0 (at least one eigenvalue = 0);
bi-linear is simple: a = 0 means degenerated to
linear anyway: no critical point at all! (except constant function)
(but with more than one cell: can have max or min at vertices)



Interlude: Implicit Function Theorem

When can I write an implicit function in $\mathbb{R}^{n+m}$ such that it is the graph of a function $f:\mathbb{R}^n \to \mathbb{R}^m$ *at least locally?*

That is: is this implicitly described function an *n*-manifold embedded in $\mathbb{R}^{n+m}$? (with local coordinates in $\mathbb{R}^n$)

$$G(f) := \{(x, f(x)) | x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^m \simeq \mathbb{R}^{n+m}$$

Theorem: if *m* x *m* Jacobian matrix is invertible

(easier for scalar field: check if gradient of f is non-zero)

See **`https://en.wikipedia.org/wiki/Implicit_function_theorem`**

General result: *constant rank theorem*

From 2D to 3D (Domain)

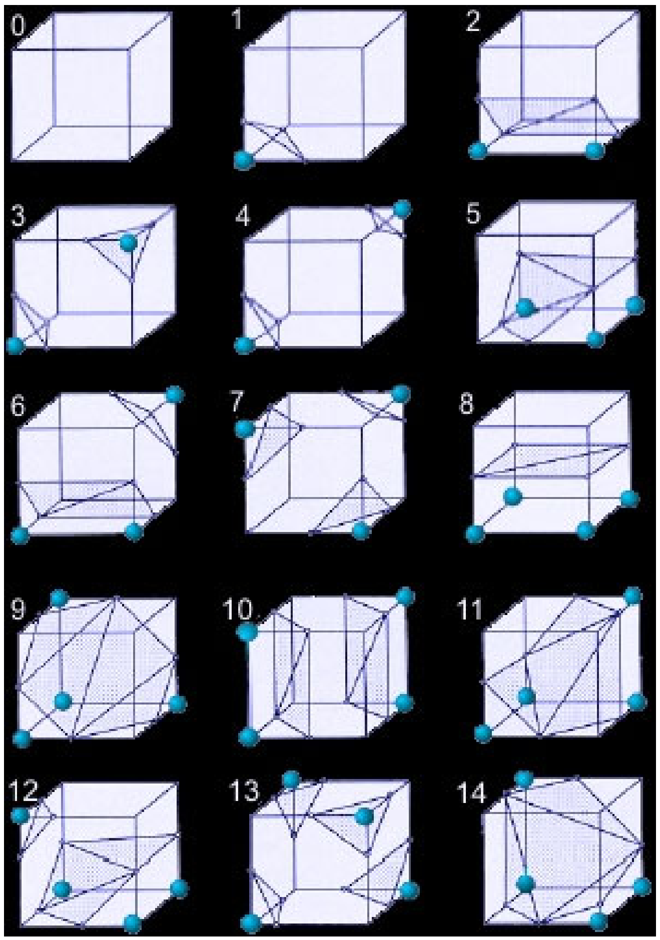
2D - Marching Squares Algorithm:

1. Locate the contour corresponding to a user-specified iso value
2. Create lines

3D - Marching Cubes Algorithm:

1. Locate the surface corresponding to a user-specified iso value
2. Create triangles
3. Calculate normals to the surface at each vertex
4. Draw shaded triangles

Marching Cubes



- For each cell, we have 8 vertices with 2 possible states each (inside or outside).
- This gives us 2^8 possible patterns = 256 cases.
- Enumerate cases to create a LUT
- Use symmetries to reduce problem from 256 to 15 cases.

Explanations

- Data Visualization book, 5.3.2
- Marching Cubes: A high resolution 3D surface construction algorithm, Lorensen & Cline, ACM SIGGRAPH 1987

The marching cubes algorithm

Contours of 3D scalar fields are known as isosurfaces.

Before 1987, isosurfaces were computed as

- contours on planar slices, followed by
- "contour stitching".

The marching cubes algorithm computes contours directly in 3D.

- Pieces of the isosurfaces are generated on a cell-by-cell basis.
- Similar to marching squares, a 8-bit number is computed from the 8 signs of $\tilde{f}(x_i)$ on the corners of a hexahedral cell.
- The isosurface piece is looked up in a table with 256 entries.

How to build up the table of 256 cases?

Lorensen and Cline (1987) exploited 3 types of symmetries:

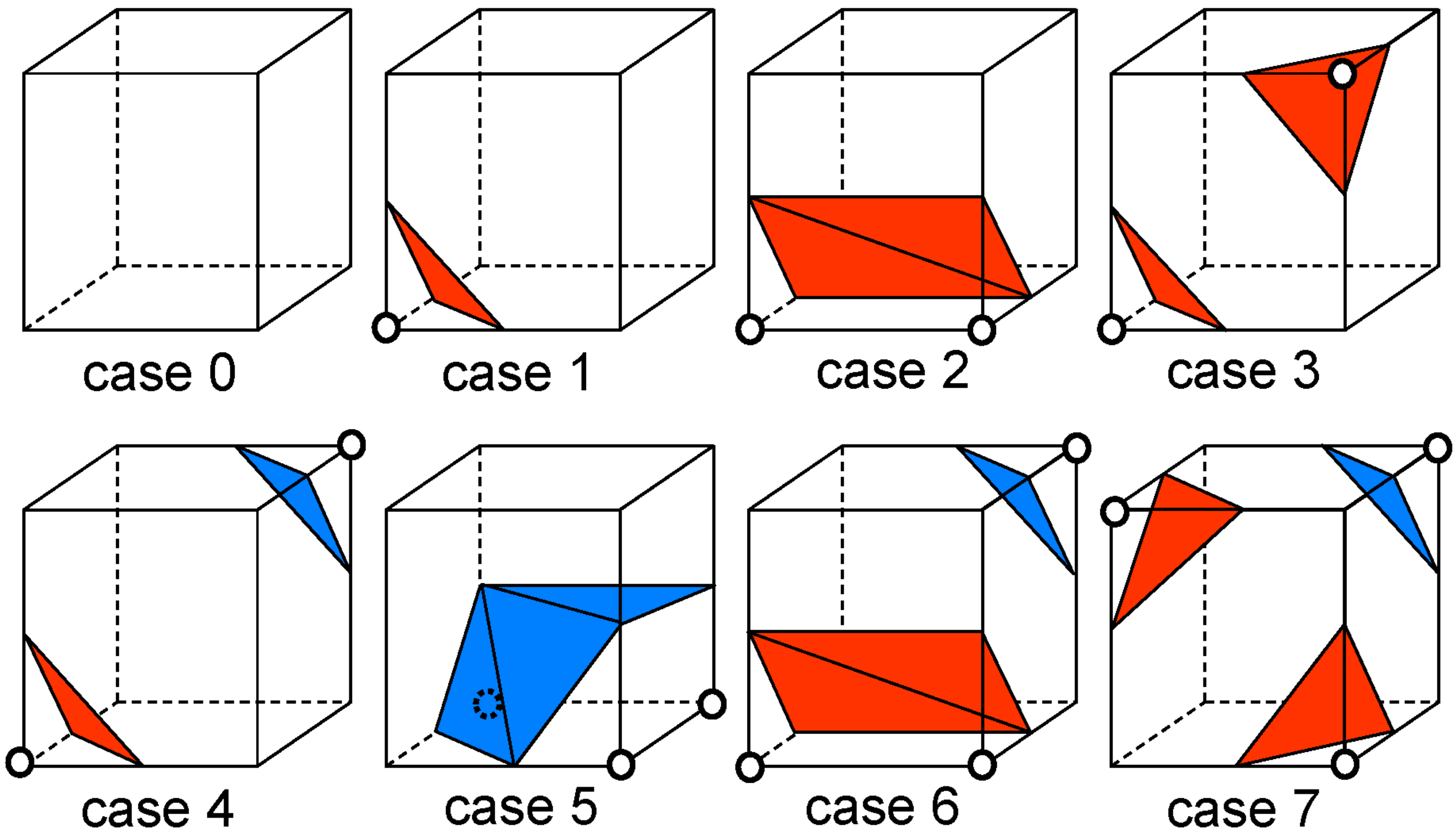
- rotational symmetries of the cube
- reflective symmetries of the cube
- sign changes of $\tilde{f}(x_i)$

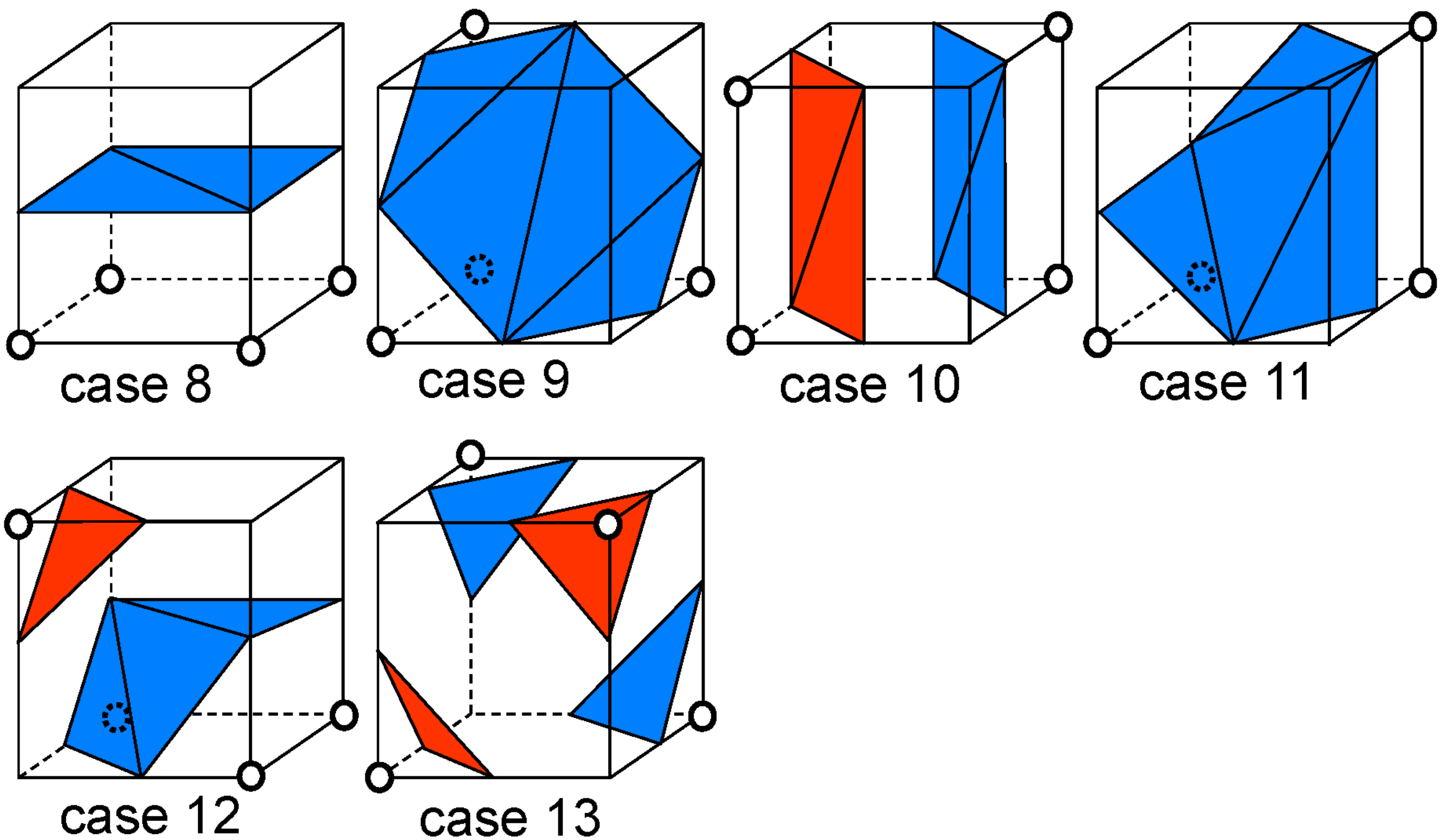
They published a reduced set of 14*) cases shown on the next slides where

- white circles indicate positive signs of $\tilde{f}(x_i)$
- the positive side of the isosurface is drawn in red, the negative side in blue.

*) plus an unnecessary "case 14" which is a symmetric image of case 11.

The marching cubes algorithm




case 8
case 9
case 10
case 11
case 12
case 13

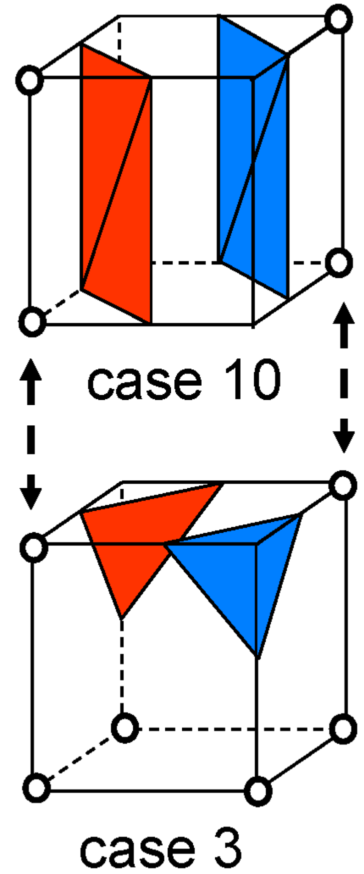
Do the pieces fit together?

- The correct isosurfaces of the trilinear interpolant would fit (trilinear reduces to bilinear on the cell interfaces)
- but the marching cubes polygons don't necessarily fit.

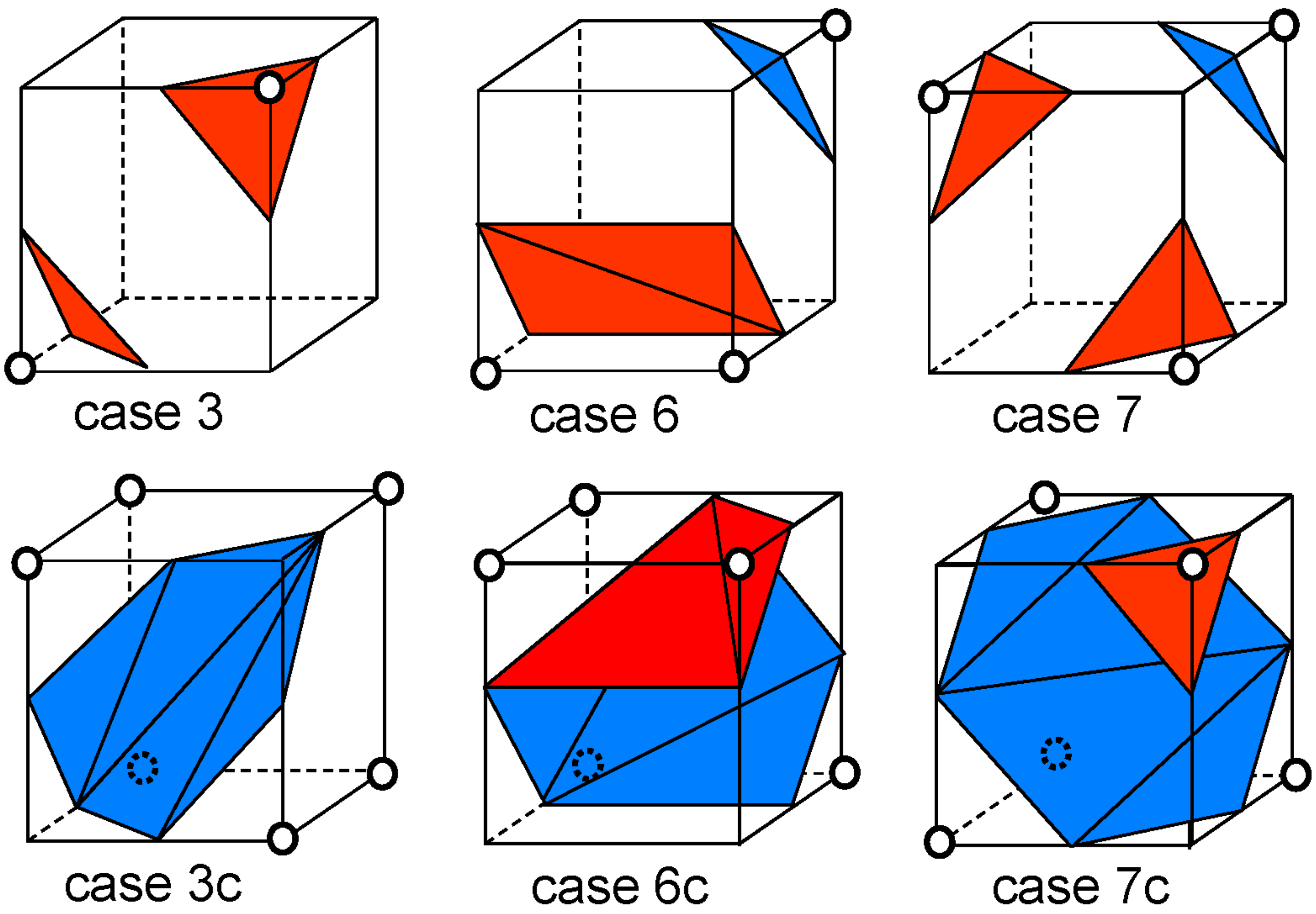
Example

- case 10, on top of
- case 3 (rotated, signs changed)

have matching signs at nodes but polygons don't fit.



The marching cubes algorithm



Summary of marching cubes algorithm:

Pre-processing steps:

- build a table of the 28 cases
- derive a table of the 256 cases, containing info on
  - intersected cell edges, e.g. for case 3/256 (see case 2/28): (0,2), (0,4), (1,3), (1,5)
  - triangles based on these points, e.g. for case 3/256: (0,2,1), (1,3,2).

The marching cubes algorithm

Loop over cells:

- find sign of $\tilde{f}(x_i)$ for the 8 corner nodes, giving 8-bit integer
- use as index into (256 case) table
- find intersection points on edges listed in table, using linear interpolation
- generate triangles according to table

Post-processing steps:

- connect triangles (share vertices)
- compute normal vectors
  - by averaging triangle normals (problem: thin triangles!)
  - by estimating the gradient of the field $f(x_i)$ (better)

Triangle Mesh Data Structure (1)

Store list of vertices; vertices shared by triangles are replicated

Render, e.g., with OpenGL immediate mode, …

coords for vertex 1

```
x1,y1,z1
x2,y2,z2
x3,y3,z3
```

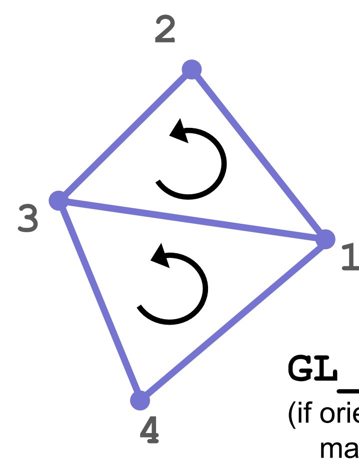
face 1

```
x1,y1,z1
x3,y3,z3
x4,y4,z4
```

face 2

```
...
```

```
struct face
  float verts[3][3]
  DataType val;
```



GL_CCW

(if orientable manifold)

Redundant, large storage size, cannot modify shared vertices easily

Store data values per face, or separately

Triangle Mesh Data Structure (2)

Indexed face set: store list of vertices; store triangles as indexes

Render using separate vertex and index arrays / buffers

coords for vertex 1

vertex list

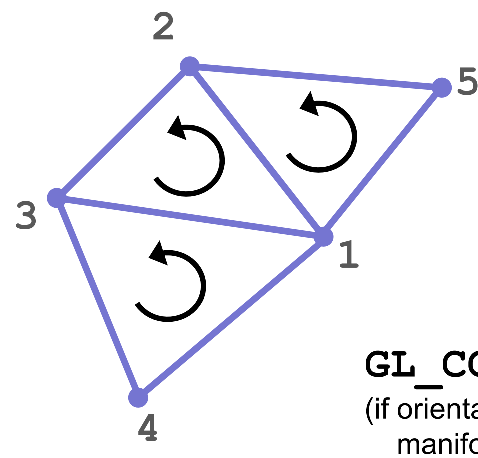
```
x1,y1,(z1)
x2,y2,(z2)
x3,y3,(z3)
x4,y4,(z4)

...
```

face list

```
1,2,3
1,3,4
2,1,5

...
```



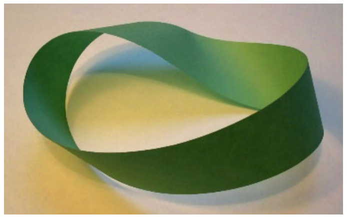
GL_CCW
(if orientable manifold)

Less redundancy, more efficient in terms of memory

Easy to change vertex positions; still have to do (global) search for shared edges (local information)

Orientability (2-manifold embedded in 3D)

not orientable



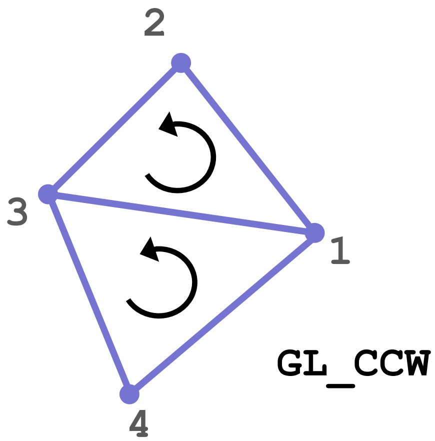
Moebius strip
(only one side!)

Orientability of 2-manifold:

Possible to assign consistent normal vector orientation

Triangle meshes

- Edges
  - Consistent ordering of vertices: CCW (counter-clockwise) or CW (clockwise)
    (e.g., (3,1,2) on one side of edge, (1,3,4) on the other side)
- Triangles
  - Consistent front side vs. back side
  - Normal vector; or ordering of vertices (CCW/CW)
  - See also: “right-hand rule”



Thank you.

Thanks for material

- Helwig Hauser
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- Philipp Muigg
- Christof Rezk-Salama