King Abdullah University of Science and Technology KAUST

CS 247 – Scientific Visualization
Lecture 25: Vector / Flow Visualization, Pt. 4

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Reading Assignment #13 (until Apr 25)

Read (required):

- Data Visualization book
  - Chapter 6.1 (Divergence and Vorticity)
- Diffeomorphisms / smooth deformations

  https://en.wikipedia.org/wiki/Diffeomorphism
- Integral curves: Stream lines, path lines, streak lines

  https://en.wikipedia.org/wiki/Integral_curve

  https://en.wikipedia.org/wiki/Streamlines,_streaklines,_and_pathlines
- Paper:
  Bruno Jobard and Wilfrid Lefer
  Creating Evenly-Spaced Streamlines of Arbitrary Density,

  http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.29.9498

Quiz #3: Apr 25

Organization

- First 30 min of lecture
- No material (book, notes, ...) allowed

Content of questions

- Lectures (both actual lectures and slides)
- Reading assignments (except optional ones)
- Programming assignments (algorithms, methods)
- Solve short practical examples

Vector fields

A static vector field $\mathbf{v}(\mathbf{x})$ is a vector-valued function of space.

A time-dependent vector field $\mathbf{v}(\mathbf{x},t)$ depends also on time.

In the case of velocity fields, the terms steady and unsteady flow are used.

The dimensions of **x** and **v** are equal, often 2 or 3, and we denote components by *x,y,z* and *u,v,w*:

$$\mathbf{x} = (x,y,z),\ \ \mathbf{v} = (u,v,w)$$

Sometimes a vector field is defined on a surface $\mathbf{x}(i,j)$. The vector field is then a function of parameters and time:

$$\mathbf{v}(i,j,t)$$

Steady vs. Unsteady Flow

- Steady flow: time-independent
  - Flow itself is static over time: $\mathbf{v}(\mathbf{x})$
  - Example: laminar flows

$$\mathbf{v}: \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \mathbf{v}(x).$$

- Unsteady flow: time-dependent
  - Flow itself changes over time: $\mathbf{v}(\mathbf{x}, t)$
  - Example: turbulent flows

$$\mathbf{v}: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad (x,t) \mapsto \mathbf{v}(x,t).$$

(here just for Euclidean domain; analogous on general manifolds)

Steady vs. Unsteady Flow

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(here just for Euclidean domain; analogous on general manifolds)

Vector fields as ODEs

For simplicity, the vector field is now interpreted as a velocity field.

Then the field $\mathbf{v}(\mathbf{x},t)$ describes the connection between location and velocity of a (massless) particle.

It can equivalently be expressed as an ordinary differential equation

$$\dot{\mathbf{x}}(t)=\mathbf{v}(\mathbf{x}(t),t)$$

This ODE, together with an initial condition

$$\mathbf{x}(t_0)=\mathbf{x}_0 ,$$

is a so-called initial value problem (IVP).

Its solution is the integral curve (or trajectory)

$$\mathbf{x}(t)=\mathbf{x}_0+\int_{t_0}^{t}\mathbf{v}(\mathbf{x}(\tau),\tau)\,d\tau$$

The integral curve is a pathline, describing the path of a massless particle which was released at time t_0 at position x_0.

Remark: $t < t_0$ is allowed.

For static fields, the ODE is autonomous:

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t))$$

and its integral curves

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(\tau))\,d\tau$$

are called field lines, or (in the case of velocity fields) streamlines.

In static vector fields, pathlines and streamlines are identical.

In time-dependent vector fields, instantaneous streamlines can be computed from a "snapshot" at a fixed time T (which is a static vector field)

$$\mathbf{v}_T(\mathbf{x}) = \mathbf{v}(\mathbf{x}, T)$$

In practice, time-dependent fields are often given as a dataset per time step. Each dataset is then a snapshot.

Streamline integration

Outline of algorithm for numerical streamline integration (with obvious extension to pathlines):

Inputs:

- static vector field $\mathbf{v}(\mathbf{x})$
- seed points with time of release $(\mathbf{x}_0, t_0)$
- control parameters:
  - step size (temporal, spatial, or in local coordinates)
  - step count limit, time limit, etc.
  - order of integration scheme

Output:

- streamlines as "polylines", with possible attributes (interpolated field values, time, speed, arc length, etc.)

Preprocessing:

- set up search structure for point location
- for each seed point:
  - global point location: Given a point **x**,
    find the cell containing **x** and the local coordinates (ξ, η, ζ)
    or ir the grid is structured:
    find the computational space coordinates $(i+\xi,\ j+\eta,\ k+\zeta)$
  - If **x** is not found in a cell, remove seed point

Integration loop, for each seed point **x**:

- interpolate ***v*** trilinearly to local coordinates (ξ,η,ζ)
- do an integration step, producing a new point **x'**
- incremental point location: For position **x'** find cell and local coordinates (ξ',η',ζ') making use of information (coordinates, local coordinates, cell) of old point **x**

Termination criteria:

- grid boundary reached
- step count limit reached
- optional: velocity close to zero
- optional: time limit reached
- optional: arc length limit reached

Integration step: widely used integration methods:

- Euler (used only in special speed-optimized techniques, e.g. GPU-based texture advection)

$$\mathbf{x}_{new} = \mathbf{x} + \mathbf{v}(\mathbf{x}, t) \cdot \Delta t$$

- Runge-Kutta, 2nd or 4th order

Higher order than 4th?

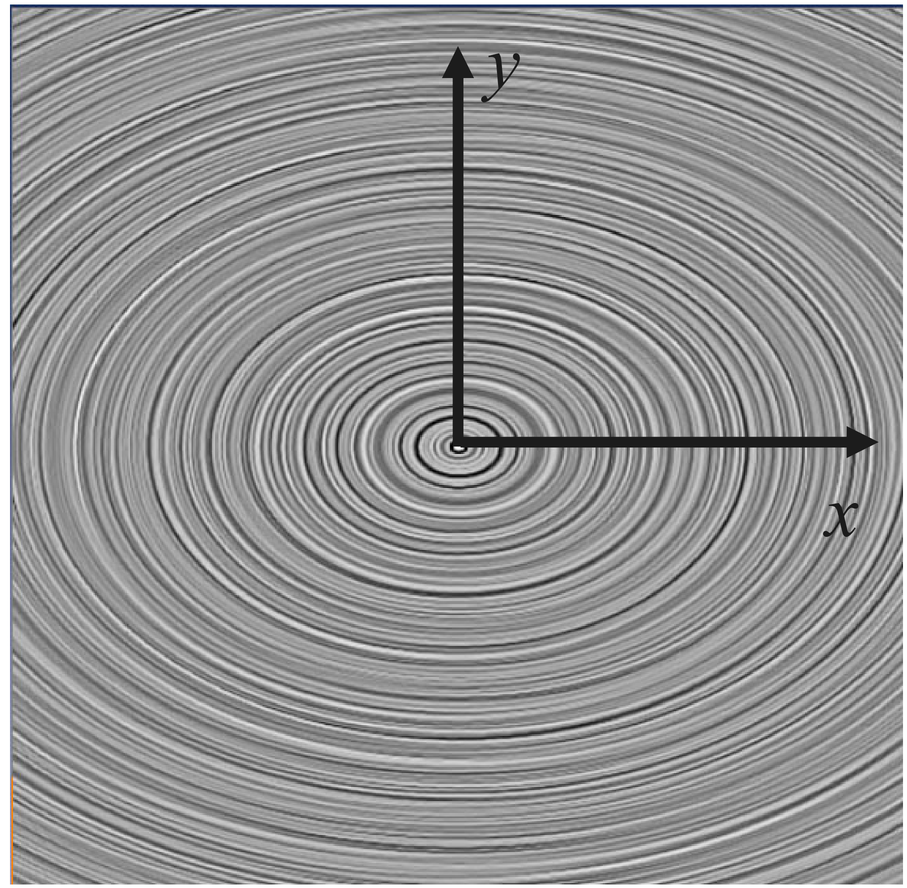
- often too slow for visualization
- study (Yeung/Pope 1987) shows that, when using standard trilinear interpolation, interpolation errors dominate integration errors.

Numerical Integration

- **Numerical integration of stream lines:**

- approximate streamline by polygon $\mathbf{x}_i$

- **Testing example:**
  - $\mathbf{v}(x,y) = (-y, x/2)$^$T$
  - exact solution: ellipses
  - starting integration from (0,-1)

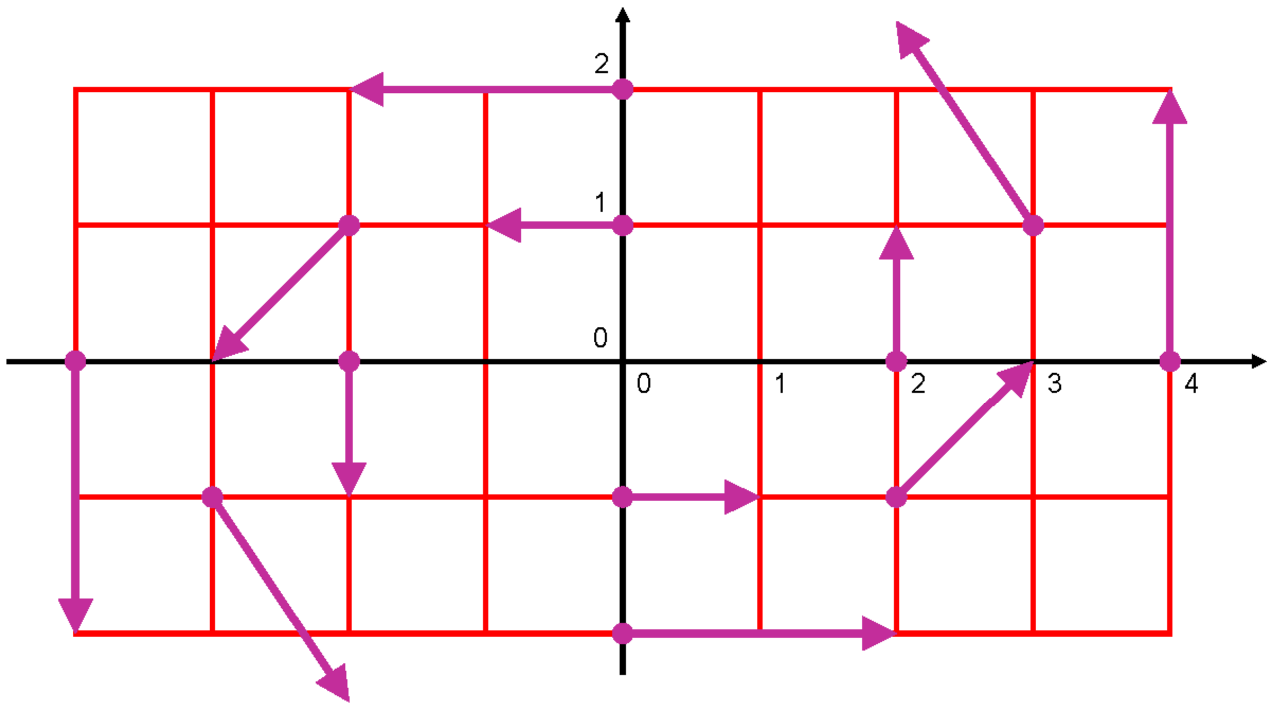


Streamlines – Practice

- Basic approach:
  - theory: $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{s}(u))\,\mathrm{d}u$
  - practice: numerical integration
  - idea:
    (very) locally, the solution is (approx.) linear
  - Euler integration:
    follow the current flow vector $\mathbf{v}(\mathbf{s}_i)$ from the current streamline point $\mathbf{s}_i$ for a very small time (dt) and therefore distance
  - Euler integration: $\mathbf{s}_{i+1} = \mathbf{s}_i + \mathrm{d}t \cdot \mathbf{v}(\mathbf{s}_i)$,
    integration of small steps (dt very small)

Euler Integration – Example

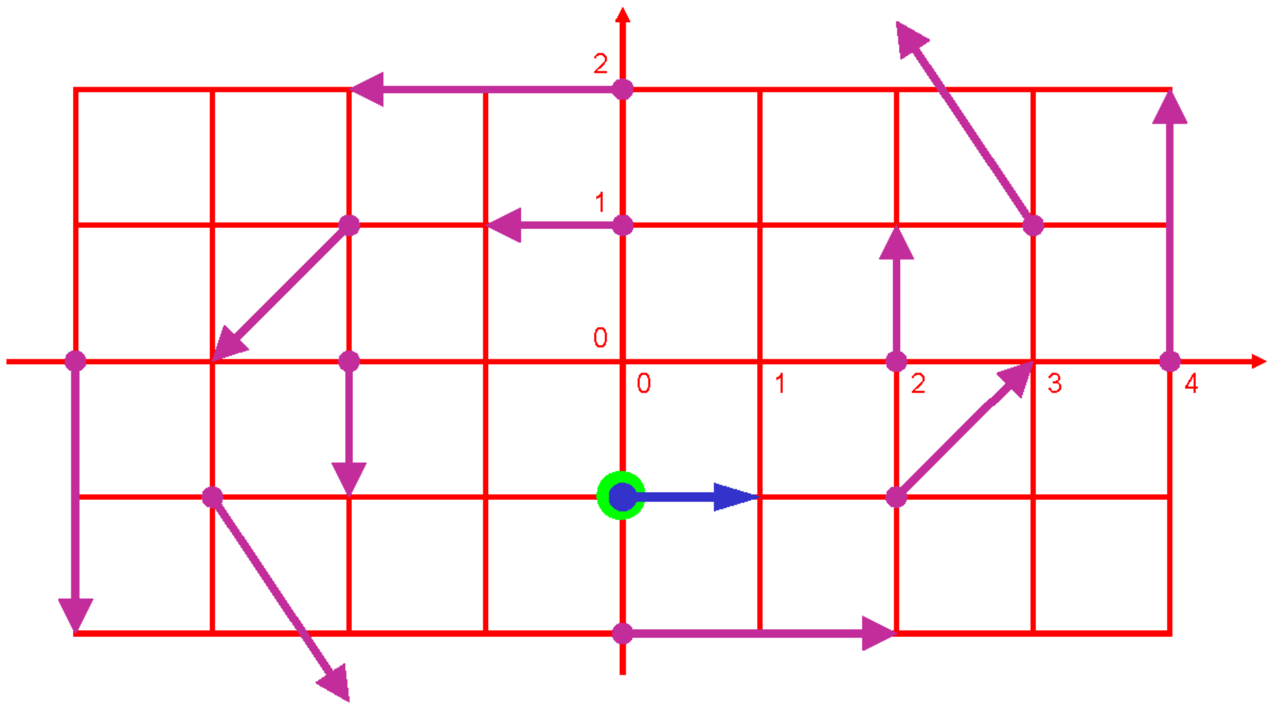
- 2D model data: $v_x = dx/dt = -y$
  $v_y = dy/dt = x/2$
- Sample arrows:



- True solution: ellipses!

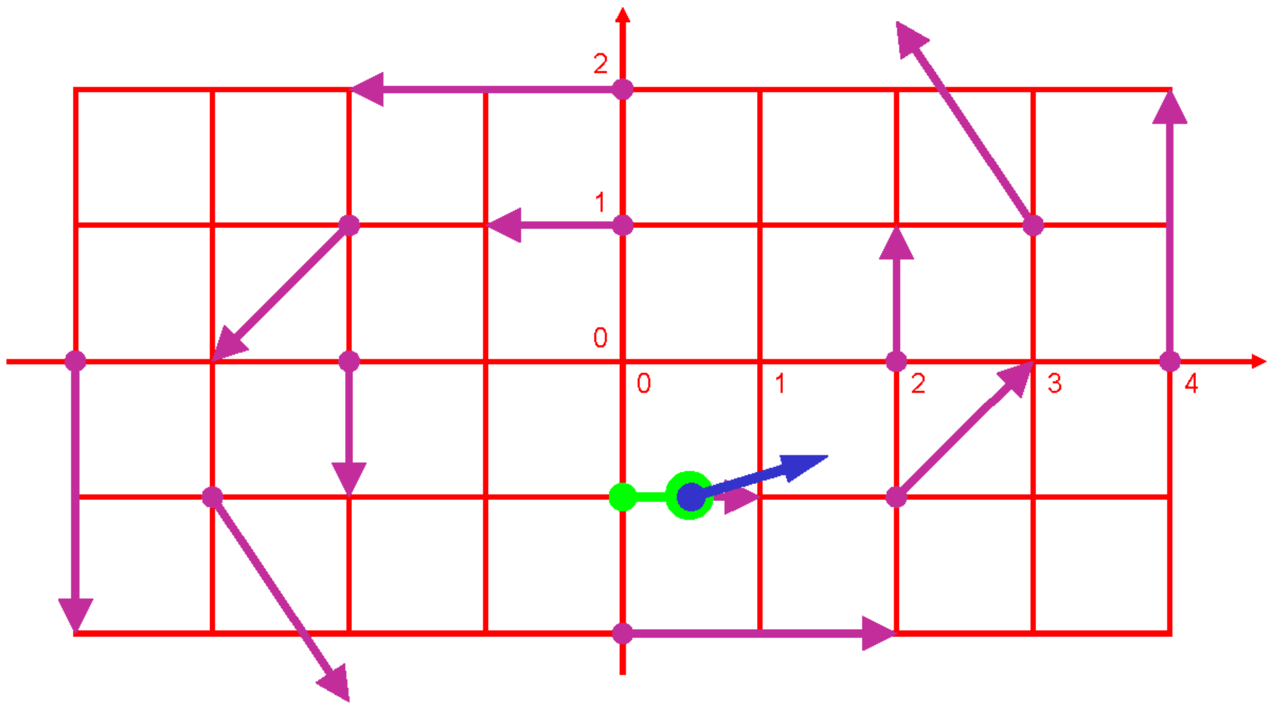
Euler Integration – Example

- Seed point $\mathbf{s}_0 = (0|-1)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_0) = (1|0)^T$;
  $dt = 1/2$



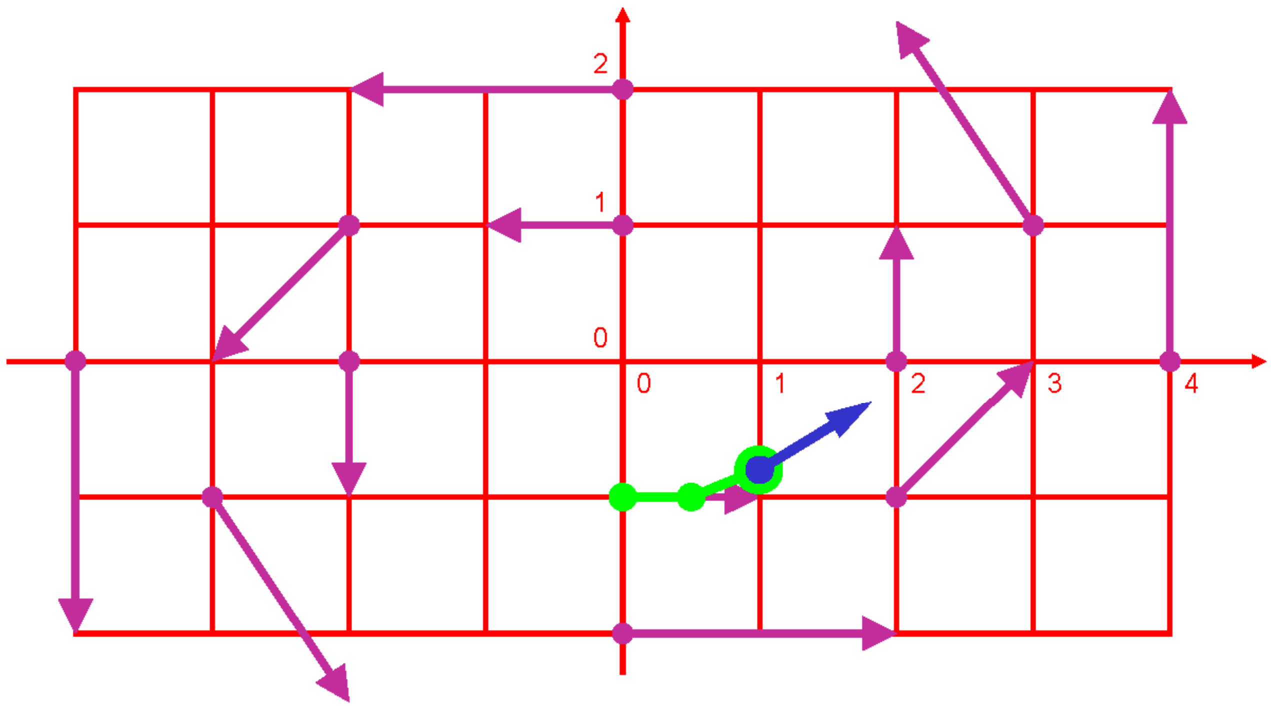
Euler Integration – Example

- New point $\mathbf{s}_1 = \mathbf{s}_0 + \mathbf{v}(\mathbf{s}_0) \cdot dt = (1/2\,|\,-1)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_1) = (1\,|\,1/4)^T$;



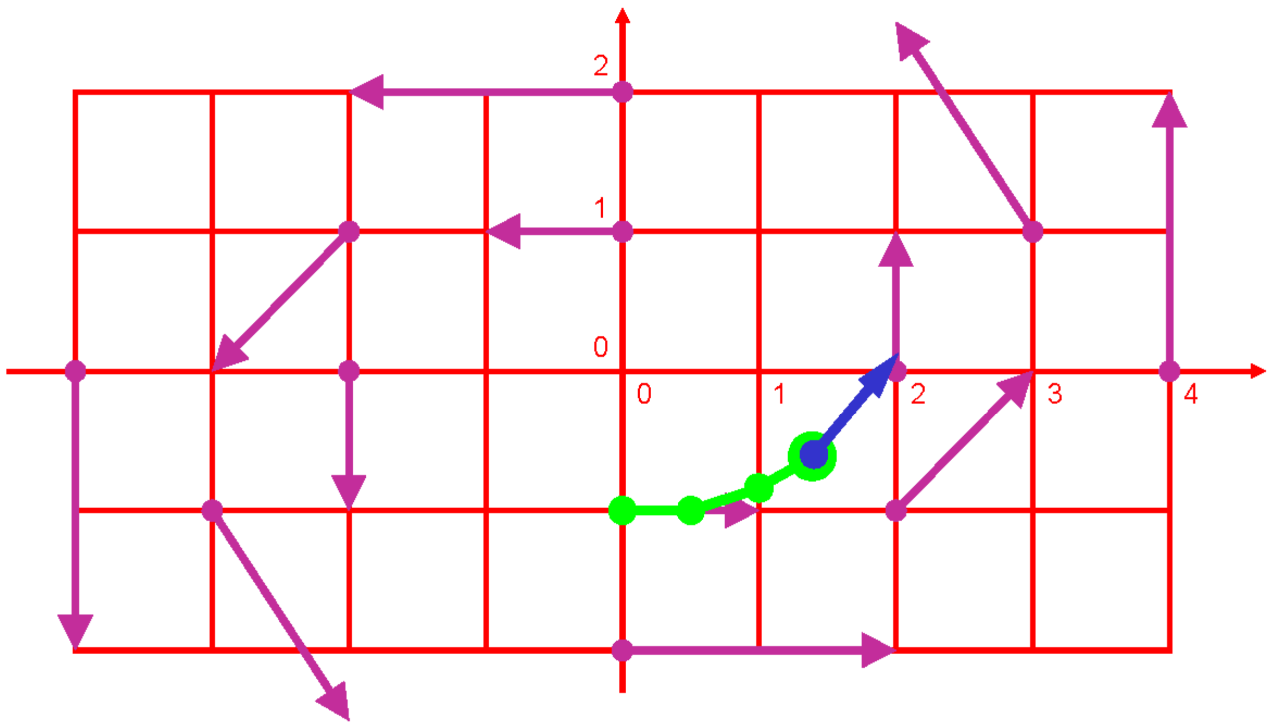
Euler Integration – Example

- New point $\mathbf{s}_2 = \mathbf{s}_1 + \mathbf{v}(\mathbf{s}_1) \cdot dt = (1\,|\,-7/8)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_2) = (7/8\,|\,1/2)^T$;



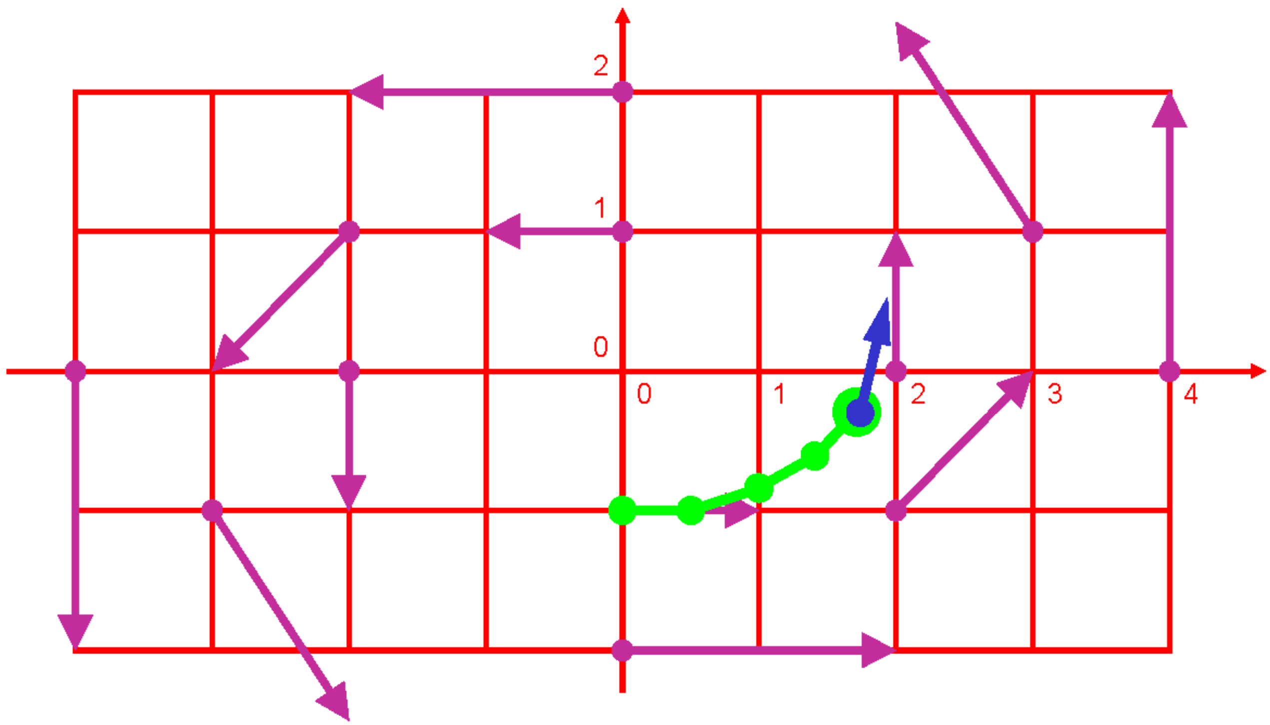
Euler Integration – Example

- $\mathbf{s}_3 = (23/16\,|\,-5/8)^T \approx (1.44\,|\,-0.63)^T;$
  $\mathbf{v}(\mathbf{s}_3) = (5/8\,|\,23/32)^T \approx (0.63\,|\,0.72)^T;$



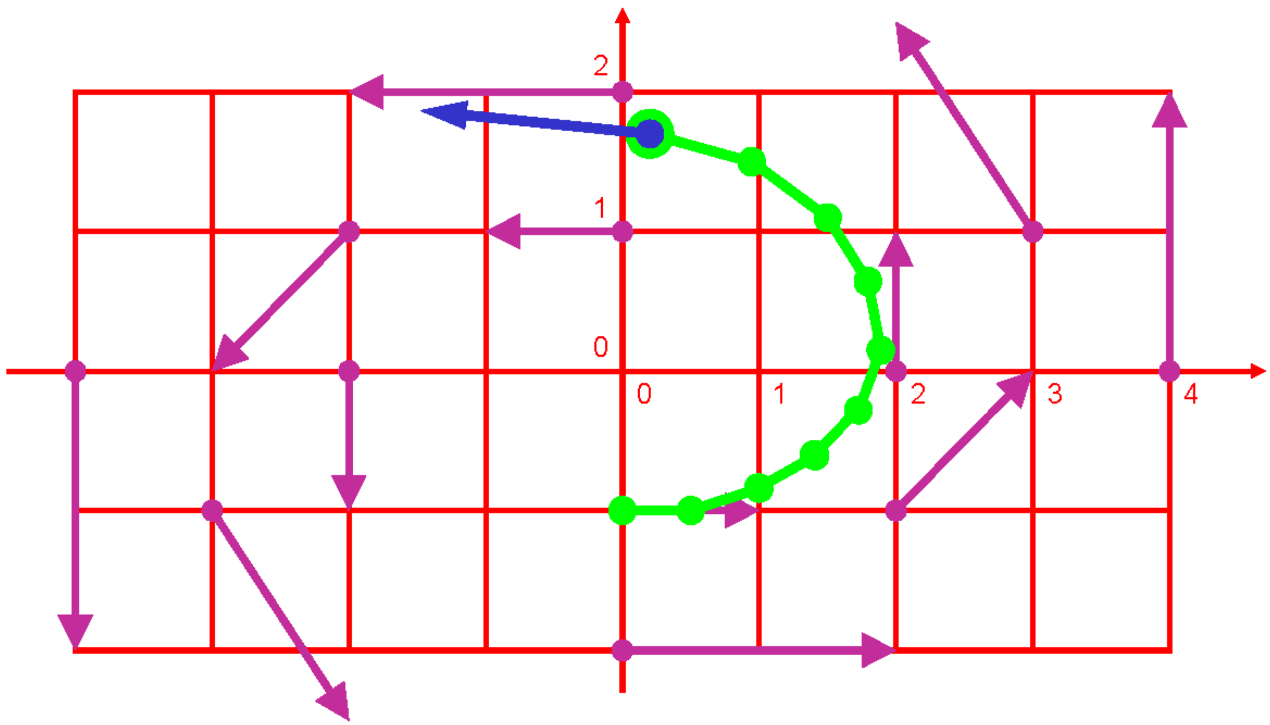
Euler Integration – Example

- $s_4 = (7/4\,|\,{-17/64})^T \approx (1.75\,|\,{-0.27})^T$;
  $v(s_4) = (17/64\,|\,7/8)^T \approx (0.27\,|\,0.88)^T$;



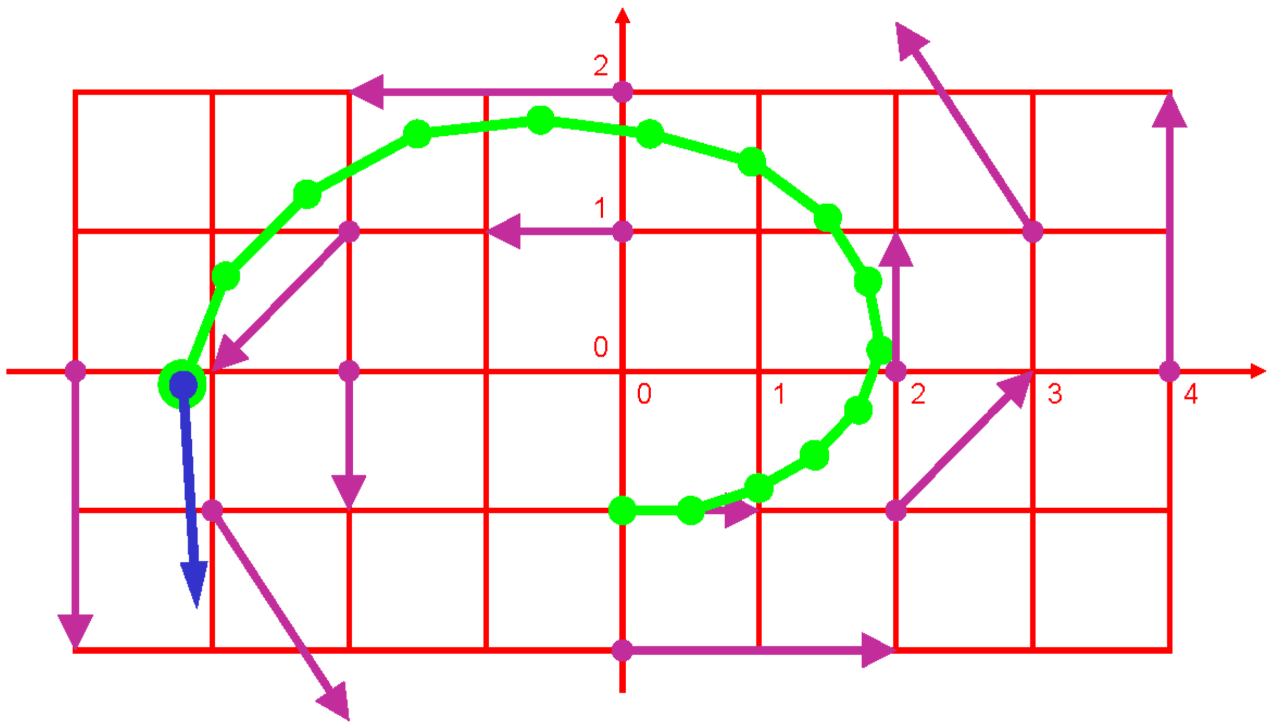
Euler Integration – Example

- $\mathbf{s}_9 \approx (0.20|1.69)^T$;
  $\mathbf{v}(\mathbf{s}_9) \approx (-1.69|0.10)^T$;



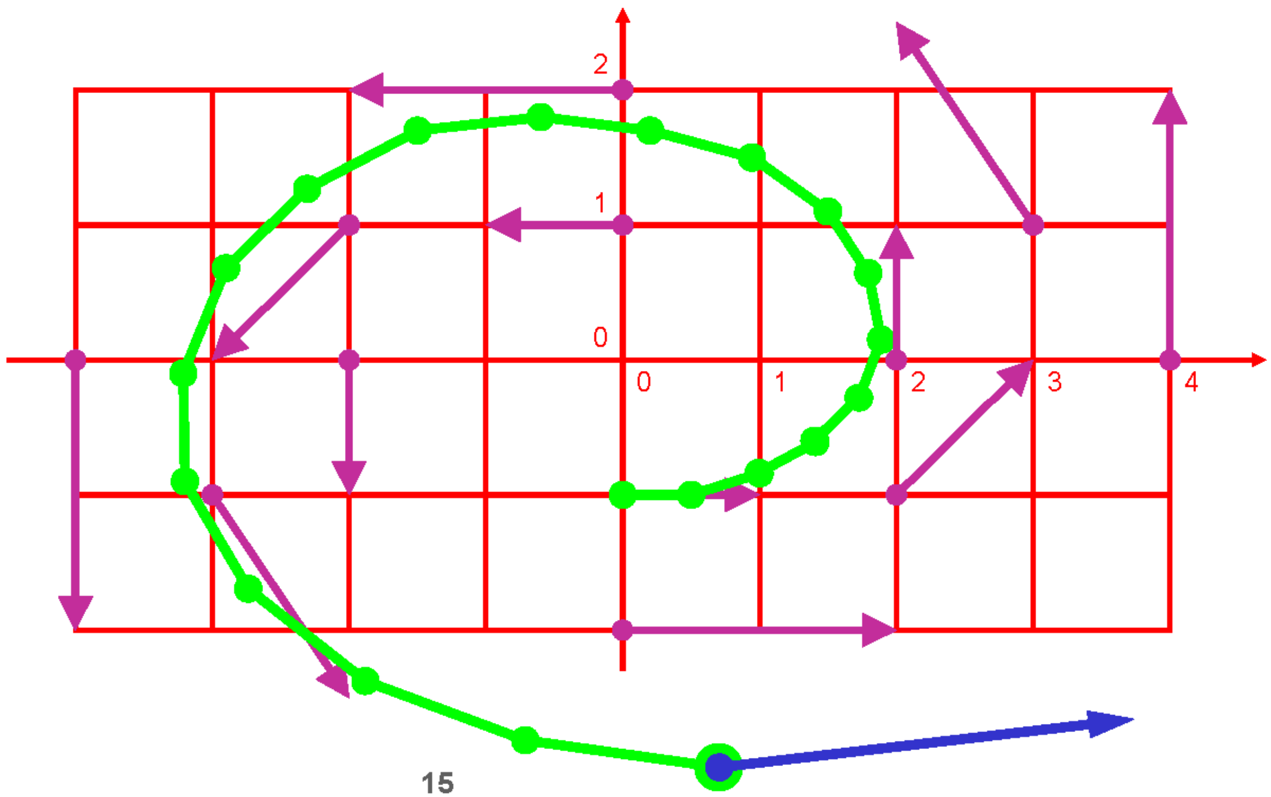
Euler Integration – Example

- $\mathbf{s}_{14} \approx (-3.22|-0.10)^T;$
  $\mathbf{v}(\mathbf{s}_{14}) \approx (0.10|-1.61)^T;$



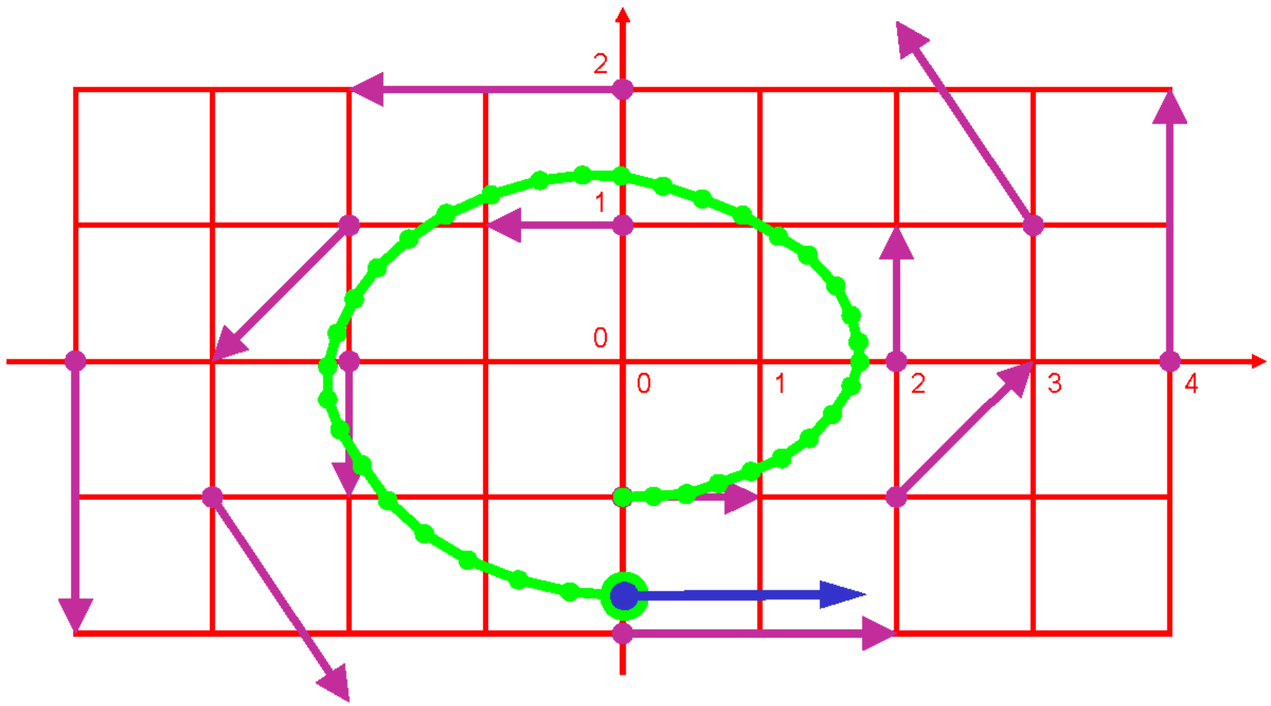
Euler Integration – Example

- $\mathbf{s}_{19} \approx (0.75|-3.02)^T$; $\mathbf{v}(\mathbf{s}_{19}) \approx (3.02|0.37)^T$;
  clearly: large integration error, d*t* too large!
  19 steps



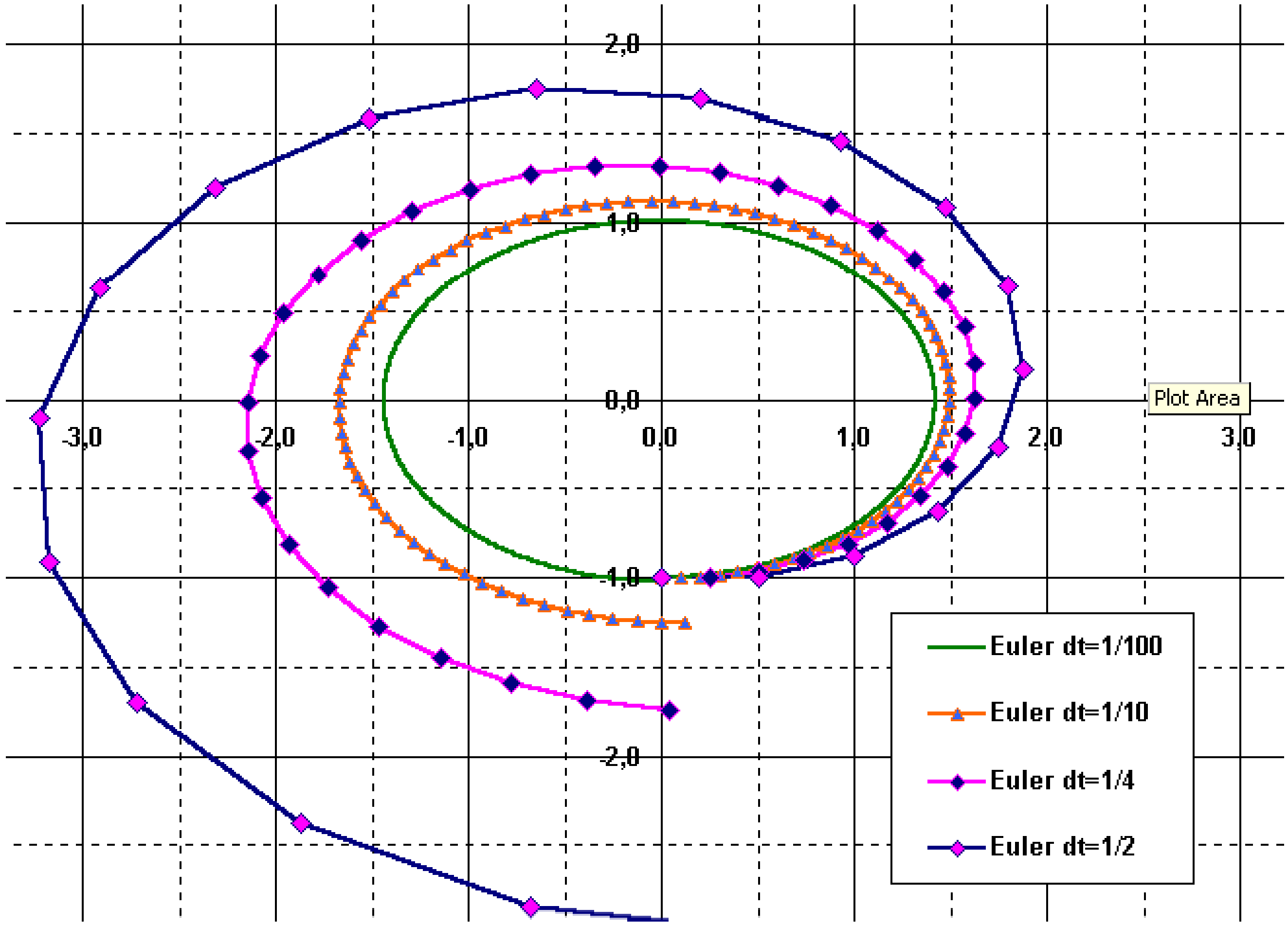
Euler Integration – Example

- d*t* smaller (1/4): more steps, more exact!
  $\mathbf{s}_{36} \approx (0.04 \,|\, -1.74)^T$; $\mathbf{v}(\mathbf{s}_{36}) \approx (1.74 \,|\, 0.02)^T$;
- 36 steps



Comparison Euler, Step Sizes

Euler
is getting
better
propor-
tionally
to d*t*

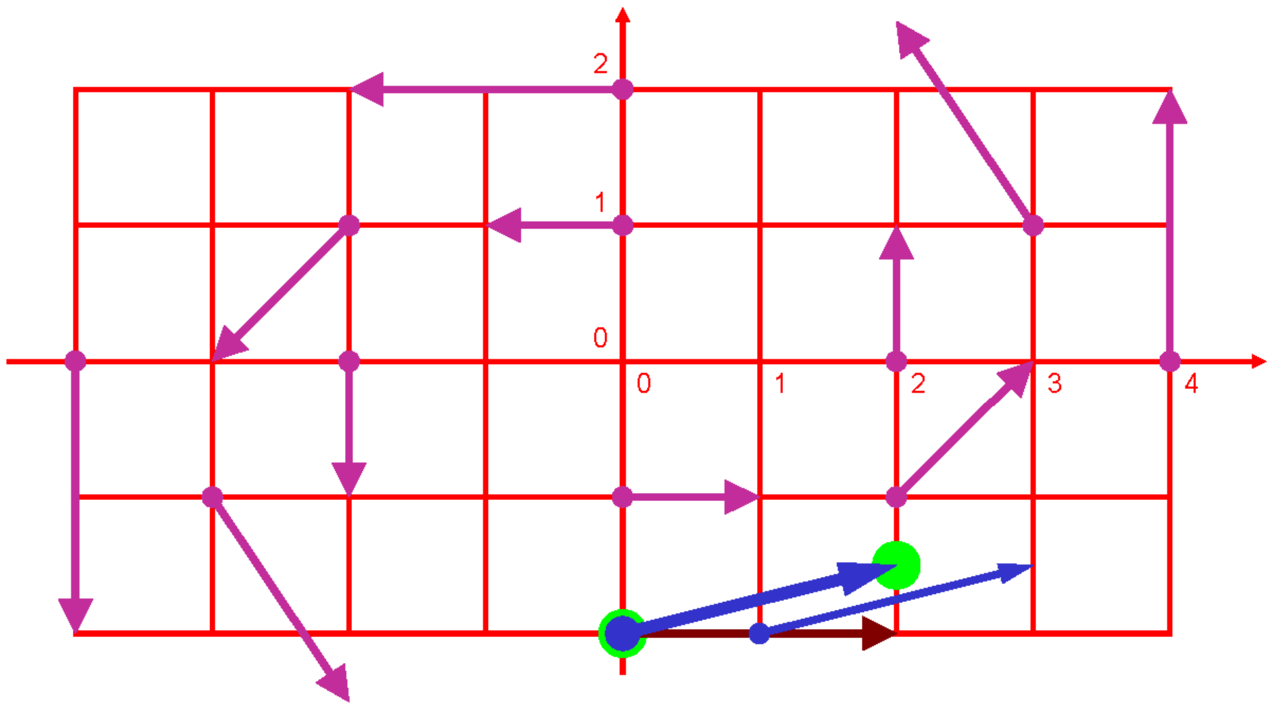


Better than Euler Integr.: RK

- Runge-Kutta Approach:
  - theory: $\mathbf{s}(t) = \mathbf{s}_0 + \int_{0 \leq u \leq t} \mathbf{v}(\mathbf{s}(u))\, du$
  - Euler: $\mathbf{s}_i = \mathbf{s}_0 + \Sigma_{0 \leq u < i} \mathbf{v}(\mathbf{s}_u) \cdot dt$
  - Runge-Kutta integration:
    - idea: cut short the curve arc
    - RK-2 (second order RK):
      1.: do half a Euler step
      2.: evaluate flow vector there
      3.: use it in the origin
    - RK-2 (two evaluations of **v** per step):
      $\mathbf{s}_{i+1} = \mathbf{s}_i + \mathbf{v}(\mathbf{s}_i + \mathbf{v}(\mathbf{s}_i) \cdot dt/2) \cdot dt$

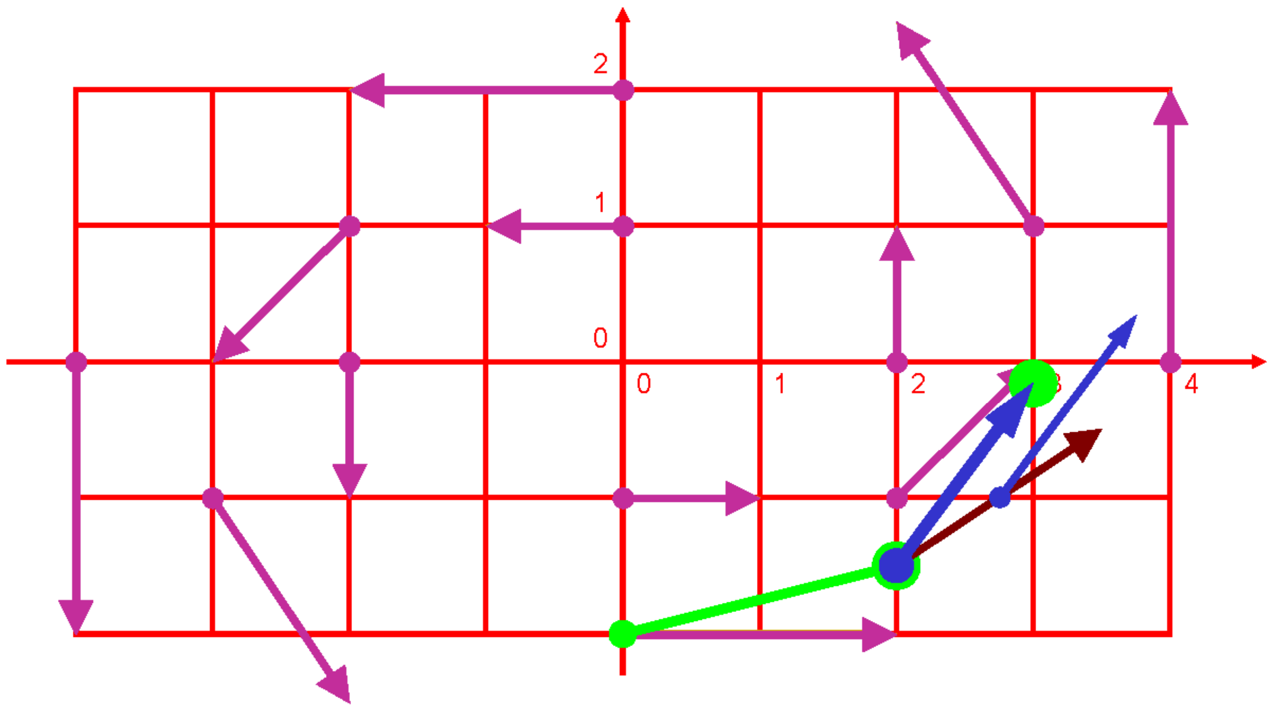
RK-2 Integration – One Step

- Seed point $\mathbf{s}_0 = (0|-2)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_0) = (2|0)^T$;
  preview vector $\mathbf{v}(\mathbf{s}_0+\mathbf{v}(\mathbf{s}_0)\cdot dt/2) = (2|0.5)^T$;
  $dt = 1$



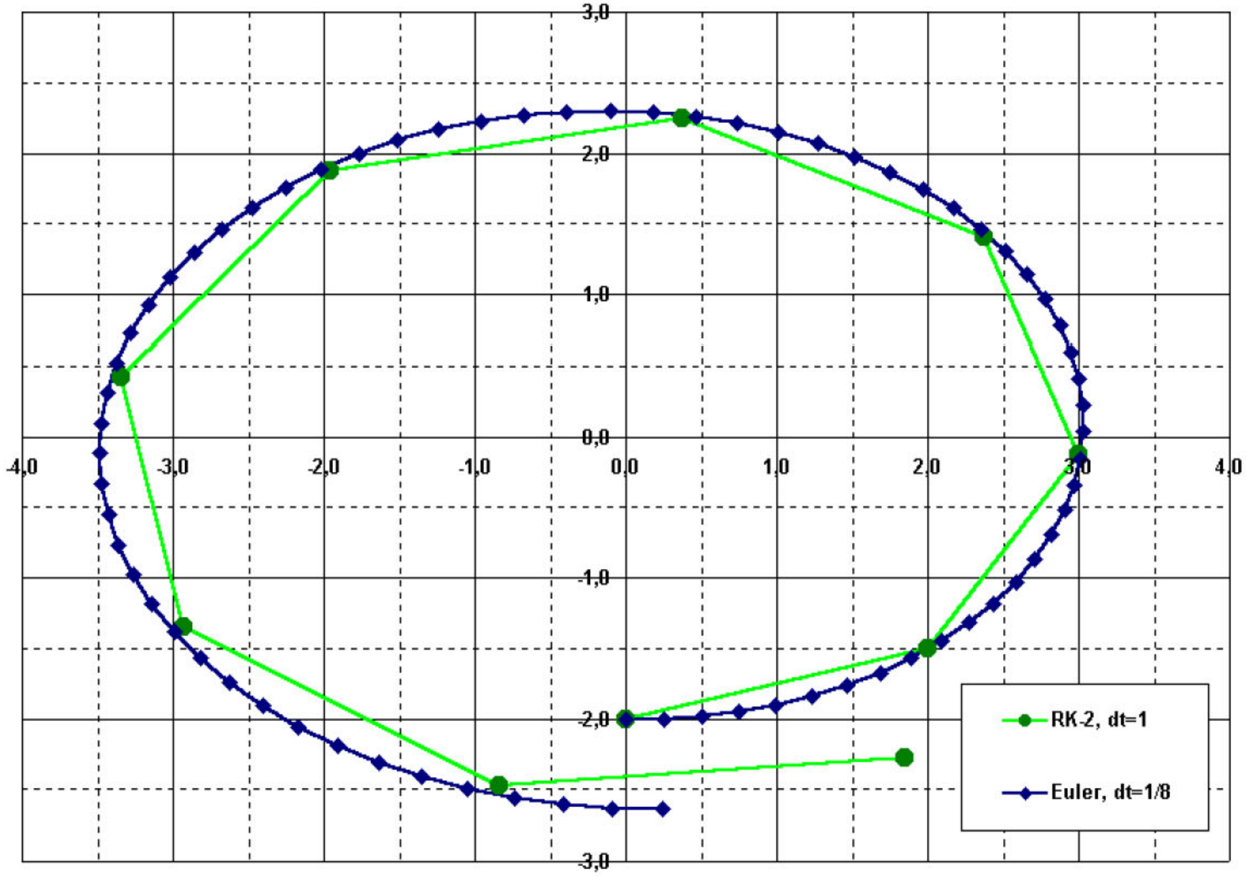
RK-2 – One more step

- Seed point $\mathbf{s}_1 = (2|-1.5)^T$;
  current flow vector $\mathbf{v}(\mathbf{s}_1) = (1.5|1)^T$;
  preview vector $\mathbf{v}(\mathbf{s}_1+\mathbf{v}(\mathbf{s}_1)\cdot dt/2) \approx (1|1.4)^T$;
  $dt = 1$



RK-2 – A Quick Round

- RK-2: even with d*t*=1 (9 steps) better than Euler with d*t*=1/8 (72 steps)

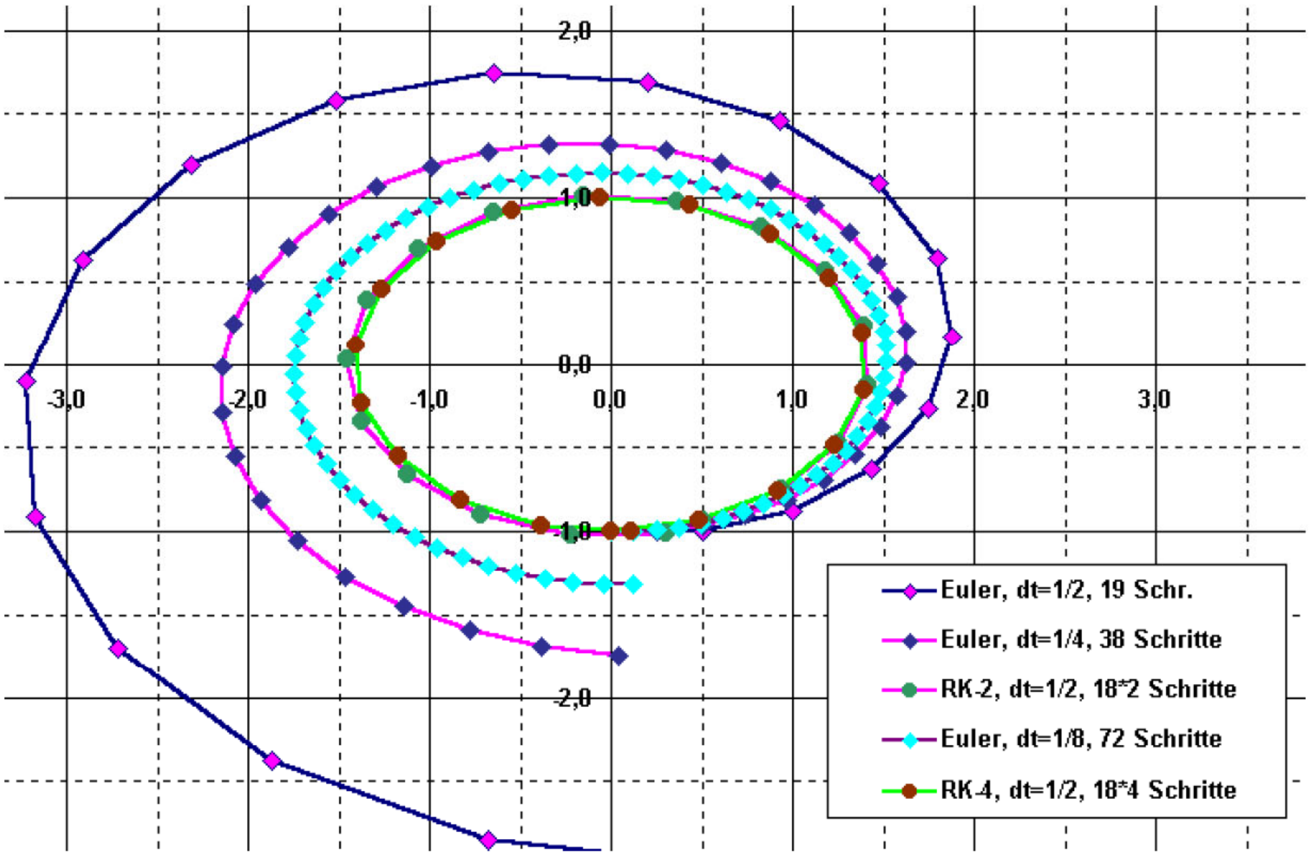


RK-4 vs. Euler, RK-2

- Even better: fourth order RK:
  - four vectors **a**, **b**, **c**, **d**
  - one step is a convex combination:
    $\mathbf{s}_{i+1} = \mathbf{s}_i + (\mathbf{a} + 2\cdot\mathbf{b} + 2\cdot\mathbf{c} + \mathbf{d})/6$
  - vectors:
    - $\mathbf{a} = \mathrm{d}t\cdot\mathbf{v}(\mathbf{s}_i)$ … original vector
    - $\mathbf{b} = \mathrm{d}t\cdot\mathbf{v}(\mathbf{s}_i+\mathbf{a}/2)$ … RK-2 vector
    - $\mathbf{c} = \mathrm{d}t\cdot\mathbf{v}(\mathbf{s}_i+\mathbf{b}/2)$ … use RK-2 …
    - $\mathbf{d} = \mathrm{d}t\cdot\mathbf{v}(\mathbf{s}_i+\mathbf{c})$ … and again!

Euler vs. Runge-Kutta

- RK-4: pays off only with complex flows
- Here approx. like RK-2



Integration, Conclusions

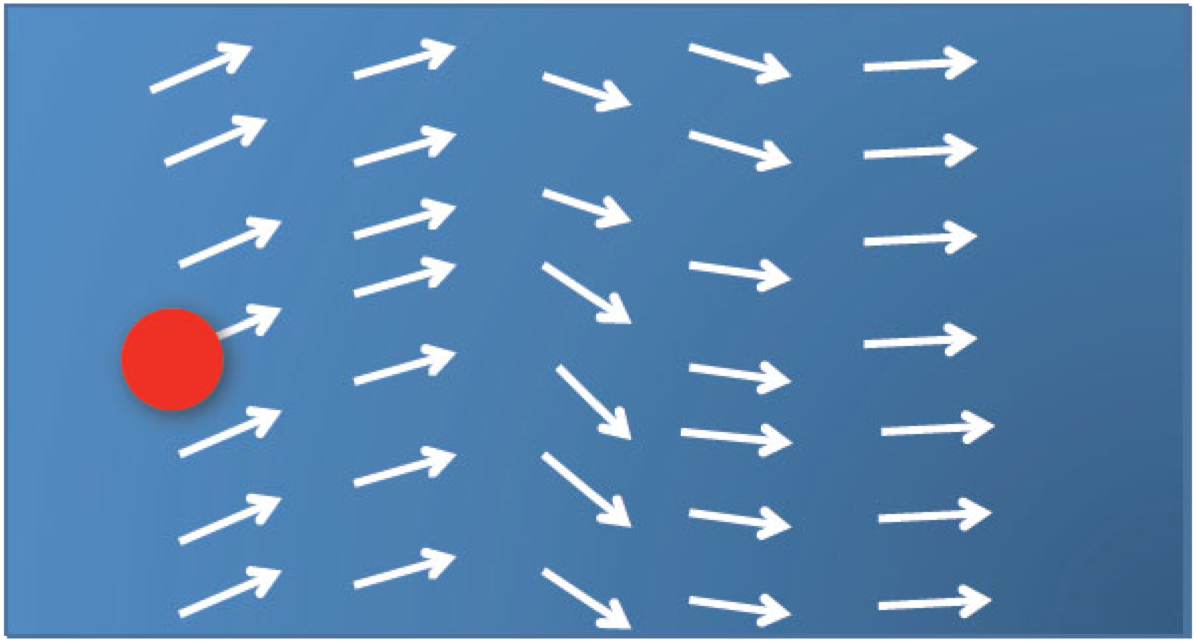
- Summary:
  - analytic determination of streamlines usually not possible
  - hence: numerical integration
  - several methods available (Euler, Runge-Kutta, etc.)
  - Euler: simple, imprecise, esp. with small dt
  - RK: more accurate in higher orders
  - furthermore: adaptive methods, implicit methods, etc.

Integral Curves, Pt. 2

Particle Trajectories

Courtesy Jens Krüger

Particle Trajectories



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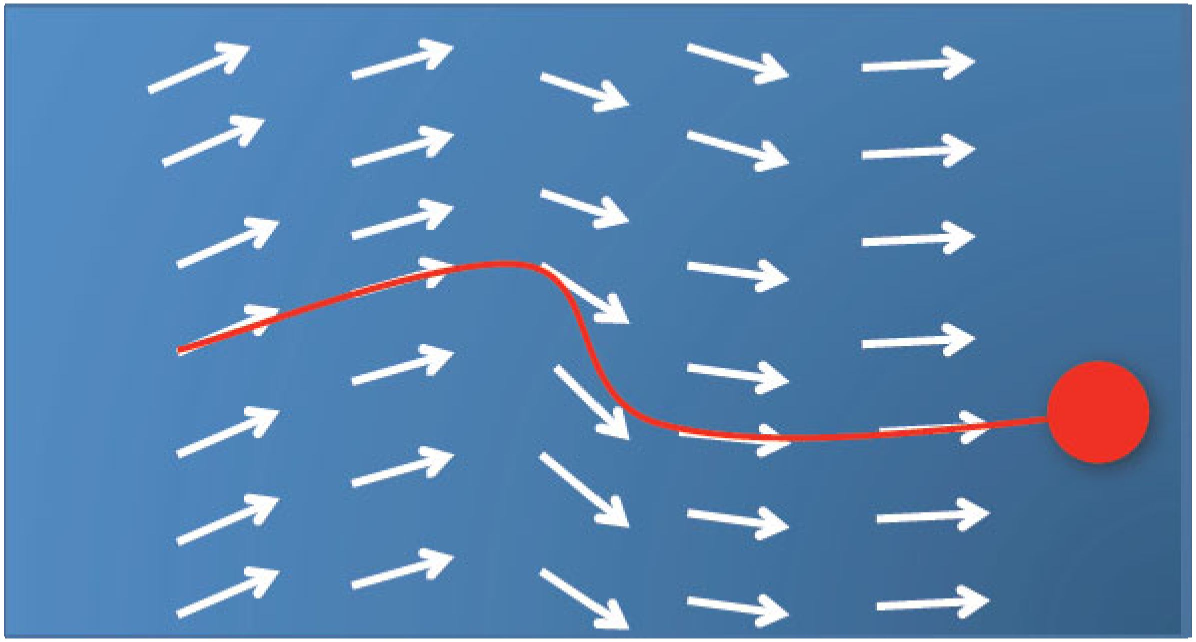
Particle Trajectories



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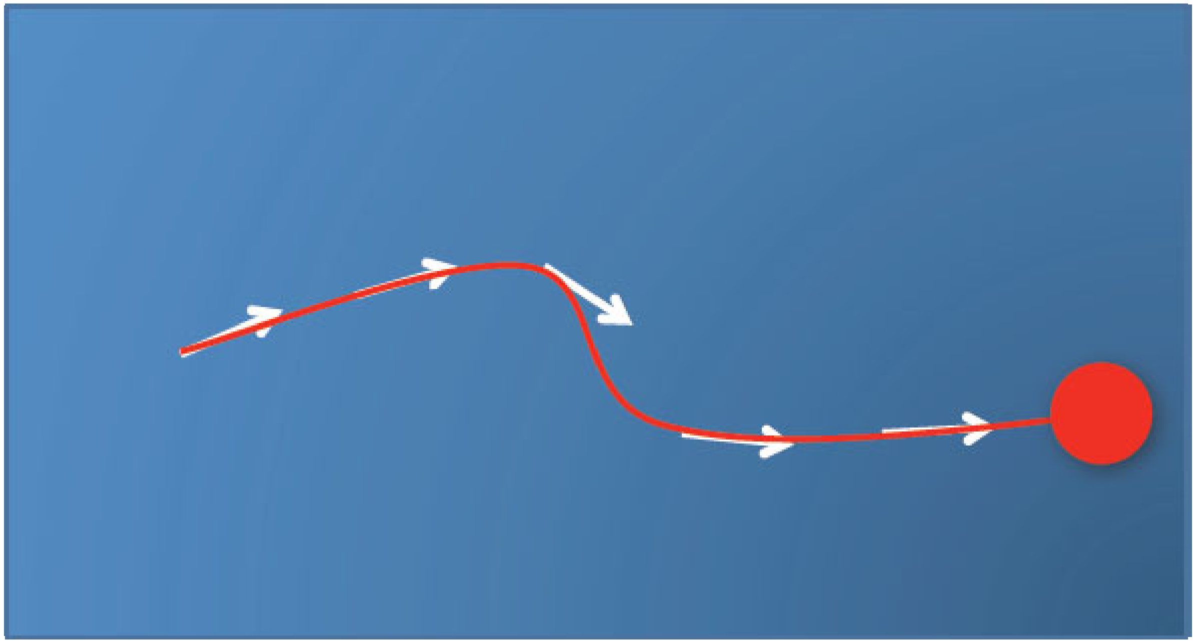

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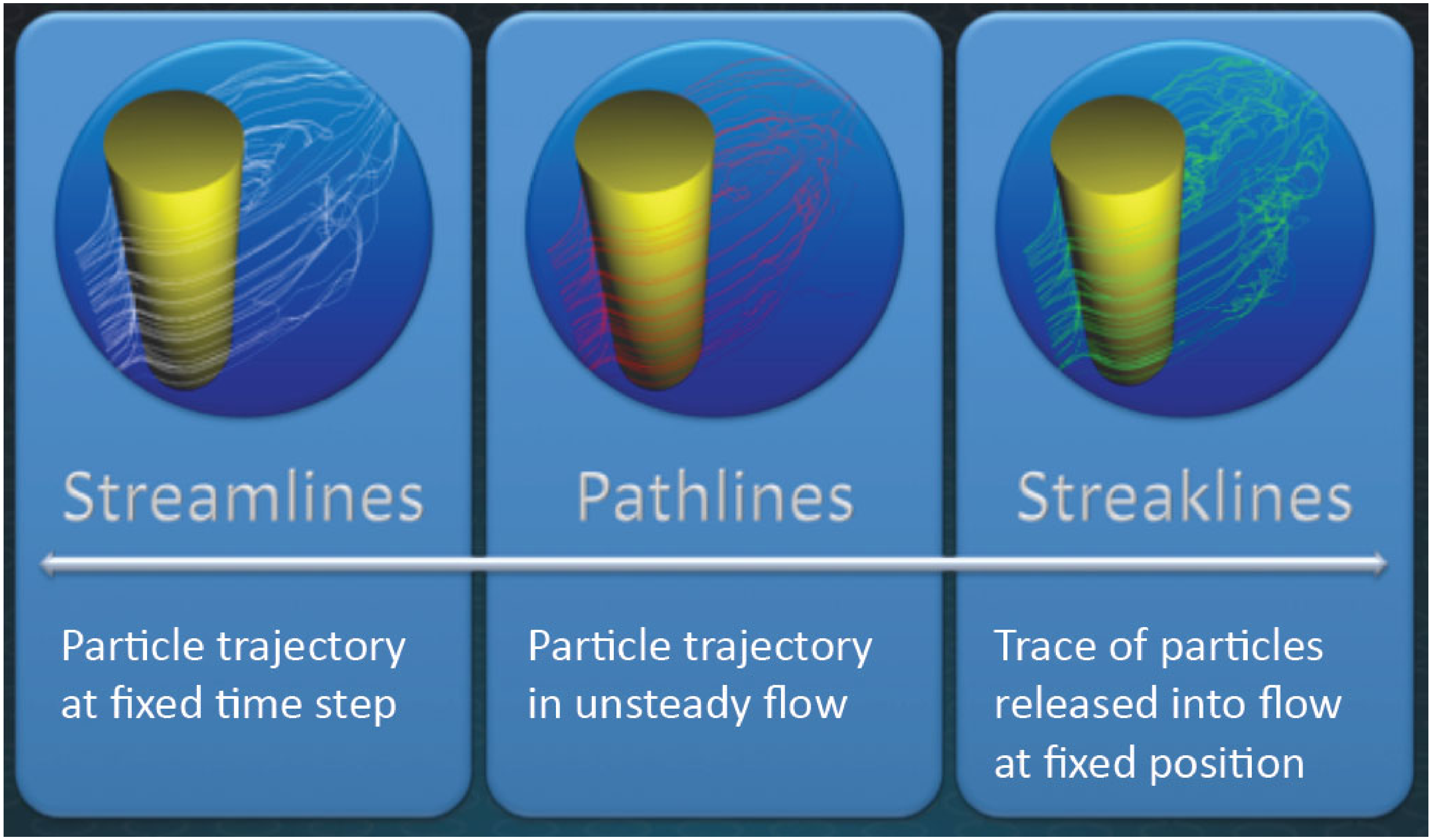
Integral Curves



Streamline

- Curve parallel to the vector field in each point for a fixed time

Pathline

- Describes motion of a massless particle over time

Streakline

- Location of all particles released at a *fixed position* over time

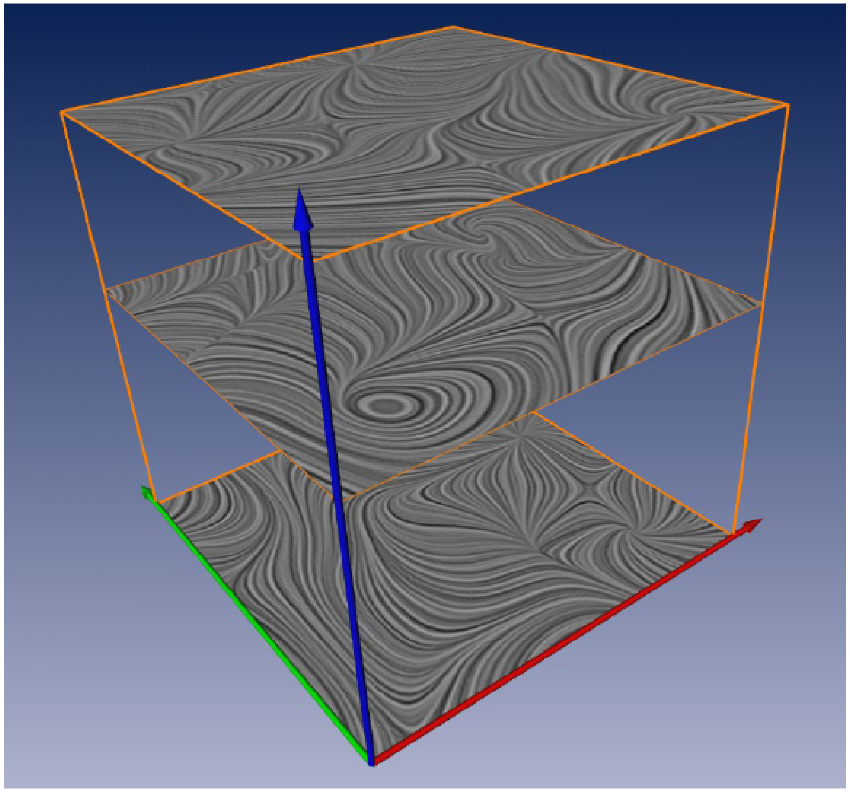
Timeline

- Location of all particles released along a line at a *fixed time*

Streamlines Over Time

Defined only for steady flow or for a fixed time step (of unsteady flow)

Different tangent curves in every time step for time-dependent vector fields (unsteady flow)

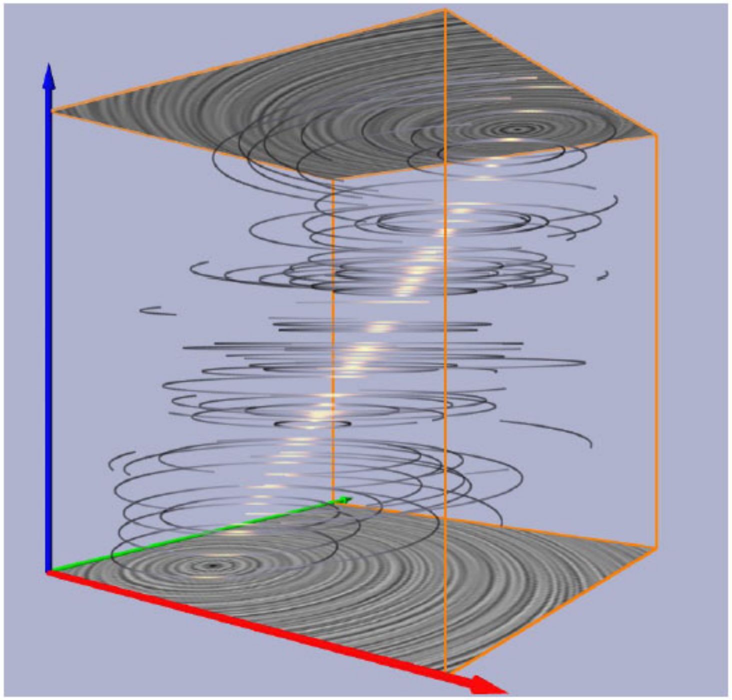


Tino Weinkauf

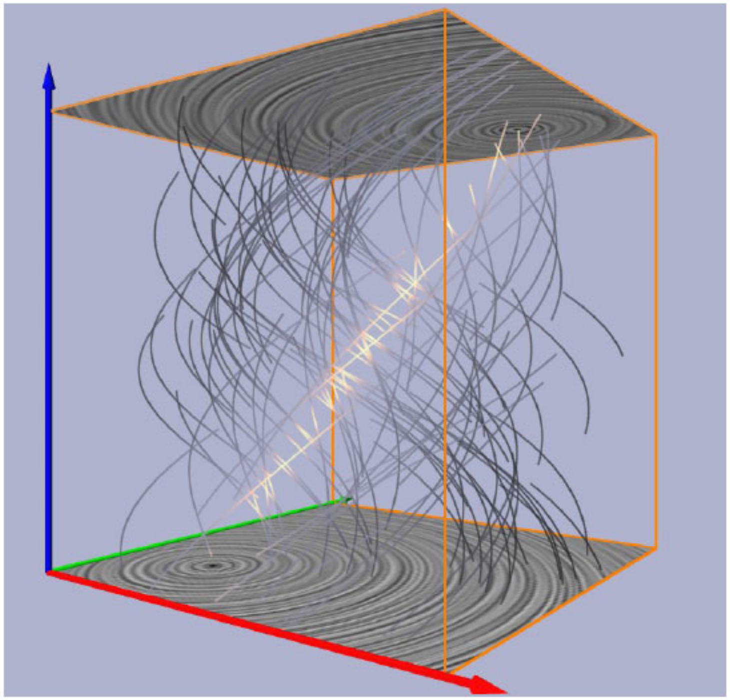
Stream Lines vs. Path Lines Viewed Over Time

Plotted with time as third dimension

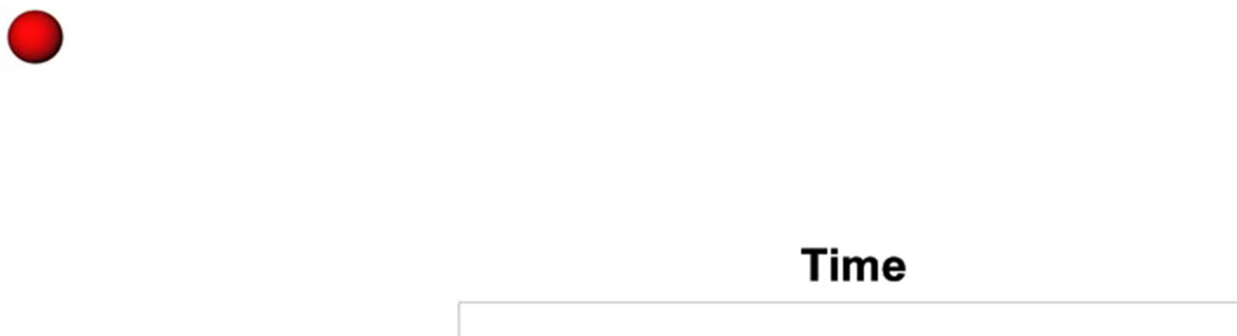
- Tangent curves to a (n + 1)-dimensional vector field



Stream Lines



Path Lines



streak line

location of all particles set out at a fixed point at different times

Particle visualization

2D time-dependent flow around a cylinder

time line

location of all particles set out on a certain line at a fixed time

The Flow / Flow Map of a Vector Field (1)

Flow of a *steady (time-independent)* vector field

- Map source position x "forward" ($t>0$) or "backward" ($t<0$) by time t

$$\boxed{\phi(x,t)} \qquad \boxed{\phi_t(x)}$$

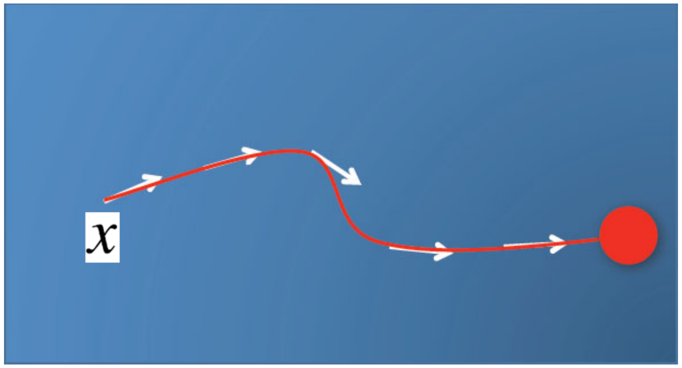
with

$$\phi_0(x) = x$$

$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

$$\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad (x,t) \mapsto \phi(x,t).$$

$$\phi_t : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \phi_t(x).$$



The Flow / Flow Map of a Vector Field (1)

Flow of a *steady (time-independent)* vector field

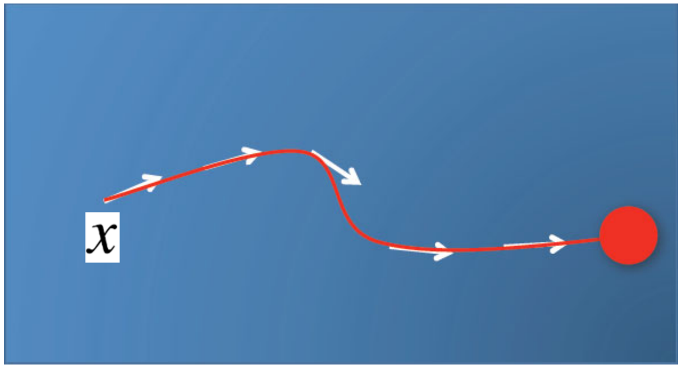
- Map source position *x* "forward" (*t*>*0*) or "backward" (*t*<*0*) by time *t*

$$\boxed{\phi(x,t)} \qquad \boxed{\phi_t(x)} \qquad \text{with} \quad \phi_0(x) = x$$

$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi(x,t).$$

$$\phi_t : M \to M, \quad x \mapsto \phi_t(x).$$



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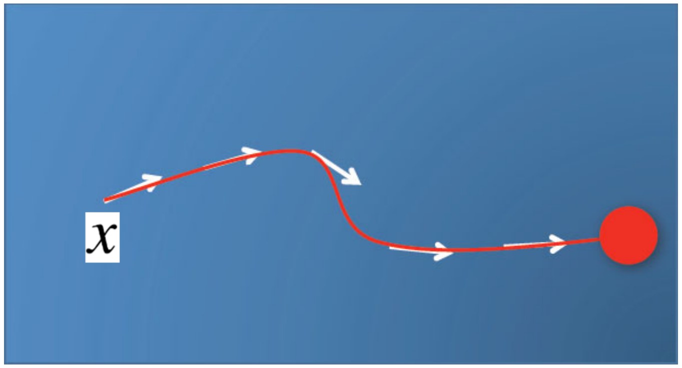
$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi(x,t). \qquad \phi_t : M \to M, \quad x \mapsto \phi_t(x).$$

$$\phi(x,t) = x + \int_0^t \mathbf{v}(\phi(x,\tau))\, \mathrm{d}\tau$$

(on a general manifold *M*, integration is performed in coordinate charts)



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Flow of a *steady (time-independent)* vector field

- Map source position *x* "forward" (*t>0*) or "backward" (*t<0*) by time *t*

$$\boxed{\phi(x,t)} \qquad \boxed{\phi_t(x)} \qquad \text{with} \quad \phi_0(x) = x$$

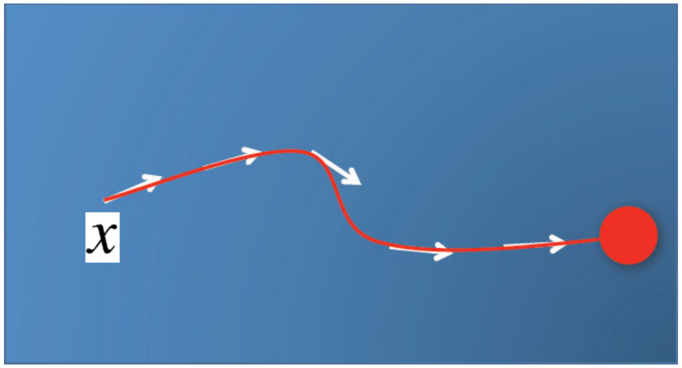
$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi(x,t). \qquad \phi_t : M \to M, \quad x \mapsto \phi_t(x).$$

- Unsteady flow? Just fix arbitrary time T

$$\phi(x,t) = x + \int_0^t \mathbf{v}(\phi(x,\tau),\mathrm{T})\,\mathrm{d}\tau$$

(on a general manifold *M*, integration is performed in coordinate charts)



The Flow / Flow Map of a Vector Field (1)

Flow of a *steady (time-independent)* vector field

- Map source position *x* "forward" (*t>0*) or "backward" (*t<0*) by time *t*

$$\boxed{\phi(x,t)} \qquad \boxed{\phi_t(x)}$$

with $\phi_0(x) = x$

$$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$$

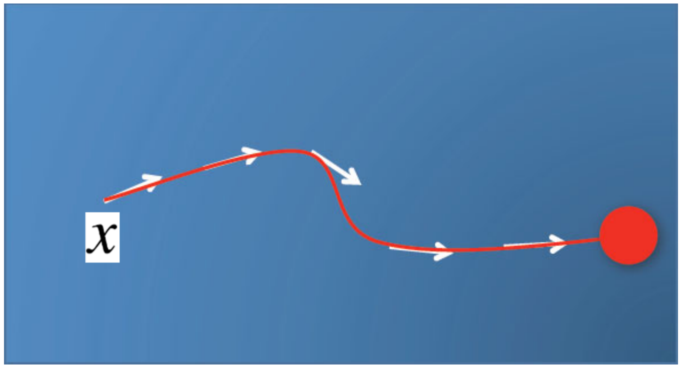
$$\phi : M \times \mathbb{R} \to M, \quad (x,t) \mapsto \phi(x,t).$$

$$\phi_t : M \to M, \quad x \mapsto \phi_t(x).$$

Can write explicitly as function of independent variable *t*, with *position x fixed*

$$t \mapsto \phi(x,t) \qquad t \mapsto \phi_t(x)$$

= stream line going through point *x*



The Flow / Flow Map of a Vector Field (2)

Flow of an *unsteady (time-dependent)* vector field

- Map source position *x* from time *s* to destination position at time *t*
  (t < s is allowed: map forward or backward in time)

$$\boxed{\psi_{t,s}(x)}$$

$$\psi_{t,s}(x) = x + \int_s^t \mathbf{v}\big(\psi_{\tau,s}(x), \tau\big)\, \mathrm{d}\tau$$

with

$$\psi_{s,s}(x) = x$$

$$\psi_{t,r}\big(\psi_{r,s}(x)\big) = \psi_{t,s}(x)$$

The Flow / Flow Map of a Vector Field (3)

Flow of an *unsteady (time-dependent)* vector field

- Map source position *x* from time *s* to destination position at time *t*
  (t < s is allowed: map forward or backward in time)

$$\boxed{\psi_{t,s}(x)} \qquad \psi_{t,s}(x) = x + \int_s^t \mathbf{v}\big(\psi_{\tau,s}(x), \tau\big)\,\mathrm{d}\tau$$

Can write explicitly as function of t, *with s and x fixed*

$t \mapsto \psi_{t,s}(x)$ → path line

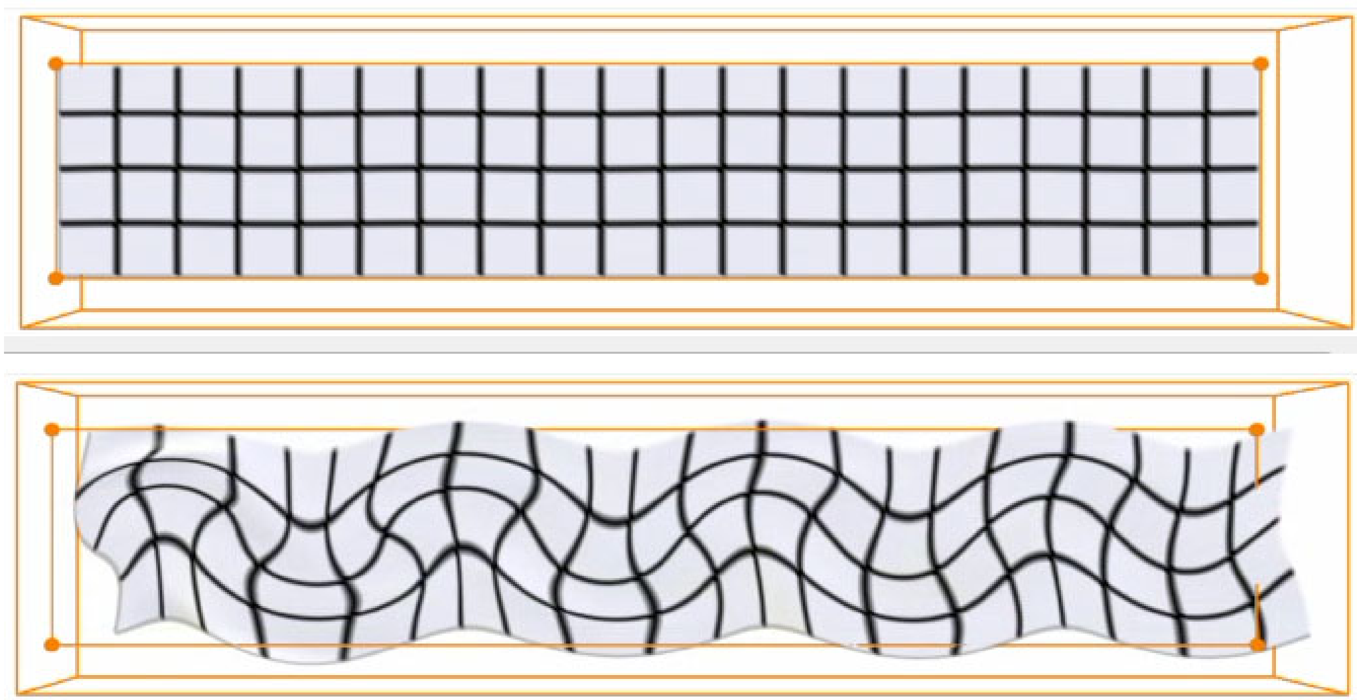
Can write explicitly as function of s, *with t and x fixed*

$s \mapsto \psi_{t,s}(x)$ → streak line

$\psi_{t,s}(x)$ is also often written as ***flow map*** $\phi_t^\tau(x)$ (with t:=s and either τ:=t or τ:=t-s)

The Flow / Flow Map of a Vector Field (4)

Can map a whole set of points (or the entire domain) through the flow map (this map is a *diffeomorphism*): $t \mapsto \psi_{t,s}(U)$



U

$= \psi_{s,s}(U)$

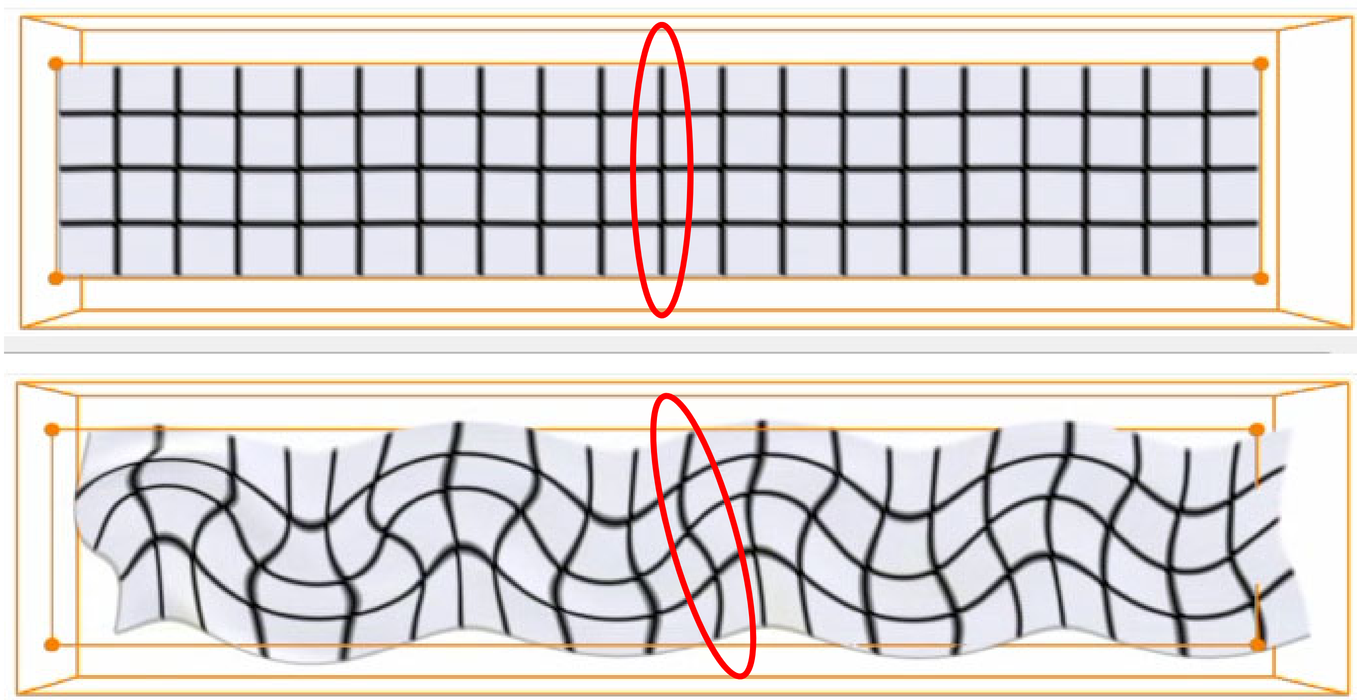
$\psi_{t,s}(U)$

(this is a *time surface!*)

The Flow / Flow Map of a Vector Field (5)

Time line: Map a whole curve from one fixed time (*s*) to another time (*t*)

$$t \mapsto \psi_{t,s}(c(\lambda))$$



$c(\lambda)$

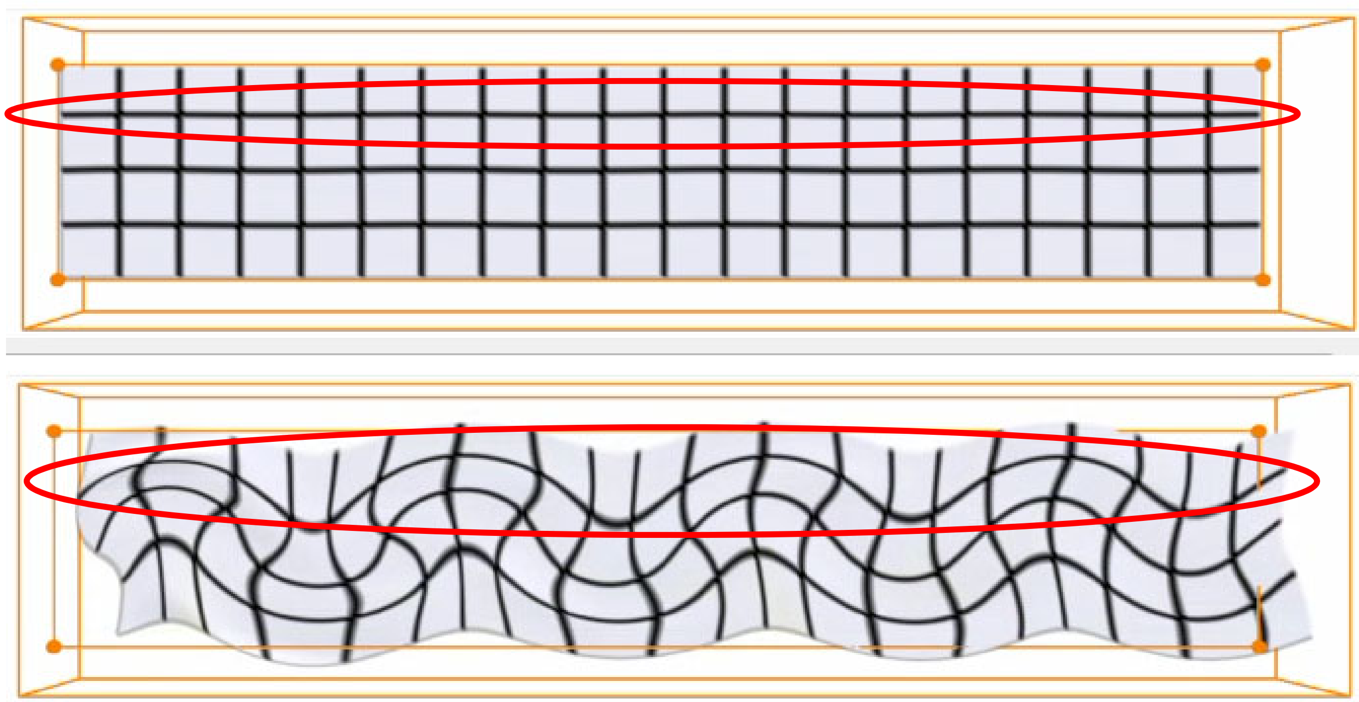
$= \psi_{s,s}(c(\lambda))$

$\psi_{t,s}(c(\lambda))$

The Flow / Flow Map of a Vector Field (5)

Time line: Map a whole curve from one fixed time (*s*) to another time (*t*)

$$t \mapsto \psi_{t,s}(c(\lambda))$$



$c(\lambda)$

$= \psi_{s,s}(c(\lambda))$

$\psi_{t,s}(c(\lambda))$

Streamline

- Curve parallel to the vector field in each point for a fixed time

Pathline

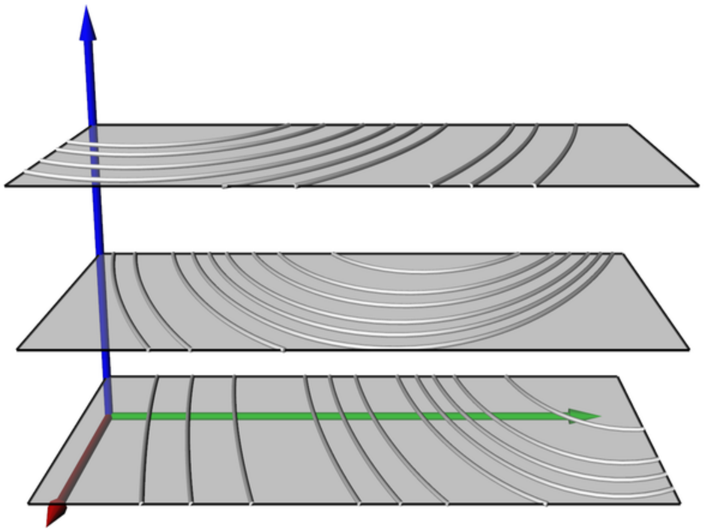
- Describes motion of a massless particle over time

Streakline

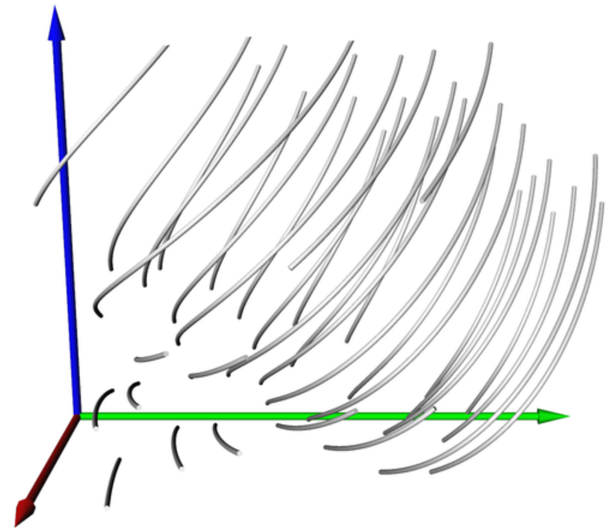
- Location of all particles released at a *fixed position* over time

Timeline

- Location of all particles released along a line at a *fixed time*

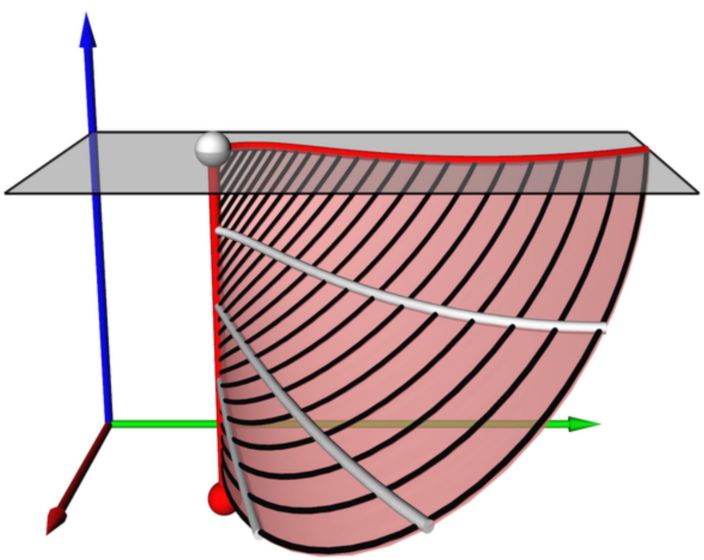


stream lines

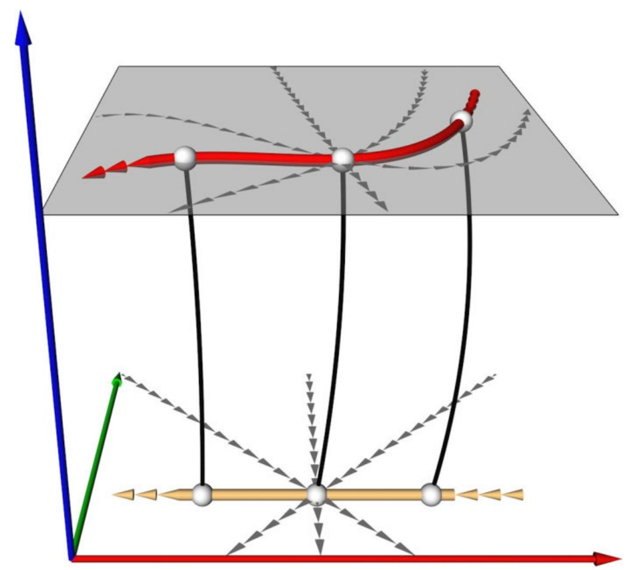


path lines

streak lines



time lines



Comparison of techniques:

(1) Pathlines:
- are physically meaningful
- allow comparison with experiment (observe marked particles)
- are well suited for dynamic visualization (of particles)

(2) Streamlines:
- are only geometrically, not physically meaningful
- are easiest to compute (no temporal interpolation, single IVP)
- are better suited for static visualization (prints)
- don't intersect (under reasonable assumptions)

(3) Streaklines:
- are physically meaningful
- allow comparison with experiment (dye injection)
- are well suited for static and dynamic visualization
- good choice for fast moving vortices
- can be approximated by set of disconnected particles

(4) Timelines:
- are physically meaningful
- are well suited for static and dynamic visualization
- can be approximated by set of disconnected particles

Thank you.

Thanks for material

- Helwig Hauser
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