King Abdullah University of Science and Technology KAUST

CS 247 – Scientific Visualization
Lecture 14: Scalar Fields, Pt. 10; Volume Rendering, Pt. 1

Markus Hadwiger, KAUST

Reading Assignment #8 (until Mar 21)

Read (required):

- Real-Time Volume Graphics, Chapter 1 (*Theoretical Background and Basic Approaches*), from beginning to 1.4.4 (inclusive)
- Real-Time Volume Graphics, Chapter 4 (Transfer Functions) until Sec. 4.4 (inclusive)

- Look at:
  Nelson Max, Optical Models for Direct Volume Rendering, IEEE Transactions on Visualization and Computer Graphics, 1995
  `http://dx.doi.org/10.1109/2945.468400`

wrapping up the previous part…

Bi-Linear Interpolation: Critical Points

Compute gradient (critical points are where gradient is zero vector):

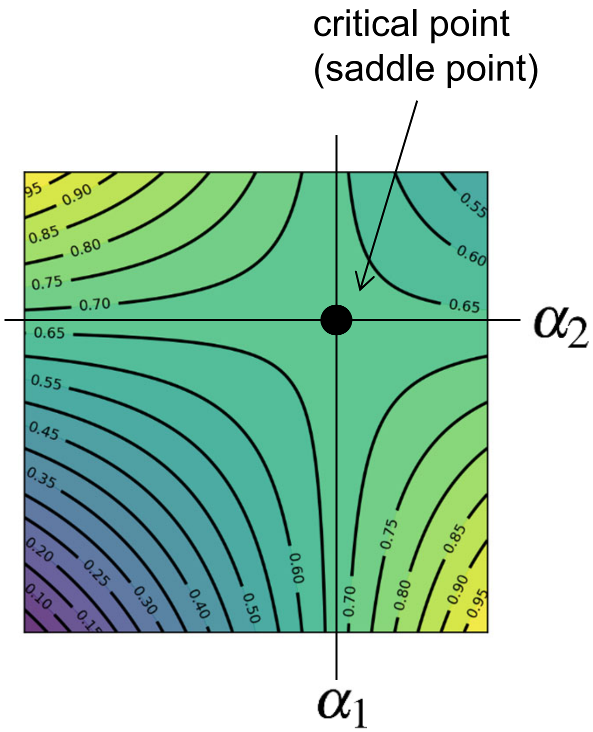
$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = (v_{10} - v_{00}) + \alpha_2(v_{00} + v_{11} - v_{10} - v_{01})$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = (v_{01} - v_{00}) + \alpha_1(v_{00} + v_{11} - v_{10} - v_{01})$$

Where are lines of constant value / critical points?

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_1} = 0 : \quad \alpha_2 = \frac{v_{00} - v_{10}}{v_{00} + v_{11} - v_{10} - v_{01}}$$

$$\frac{\partial f(\alpha_1,\alpha_2)}{\partial \alpha_2} = 0 : \quad \alpha_1 = \frac{v_{00} - v_{01}}{v_{00} + v_{11} - v_{10} - v_{01}}$$



Bi-Linear Interpolation: Critical Points

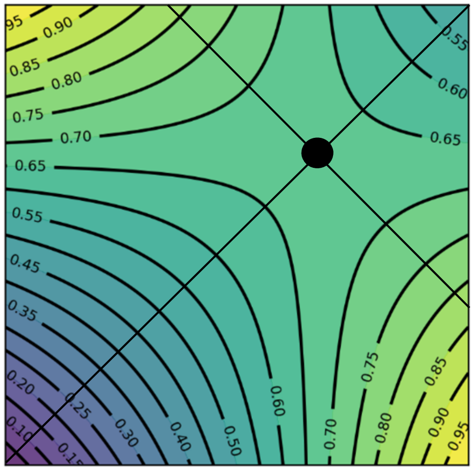
Examine Hessian matrix at critical point (non-degenerate critical p.?, ...)

$$\begin{bmatrix} \frac{\partial^2 f}{\partial \alpha_1^2} & \frac{\partial^2 f}{\partial \alpha_1 \partial \alpha_2} \\ \frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 f}{\partial \alpha_2^2} \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \qquad a = v_{00} + v_{11} - v_{10} - v_{01}$$

Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



(here also: principal curvature magnitudes and directions
of this function's graph == surface embedded in 3D)

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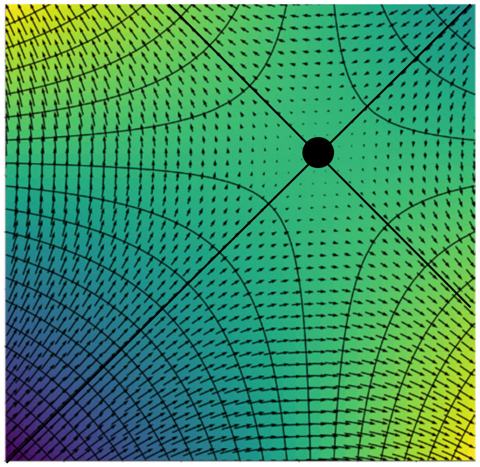

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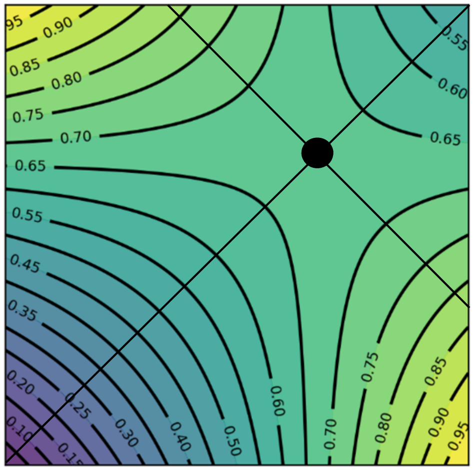
Eigenvalues and eigenvectors (Hessian is symmetric: always real)

$$\lambda_1 = -a \text{ and } \lambda_2 = a$$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

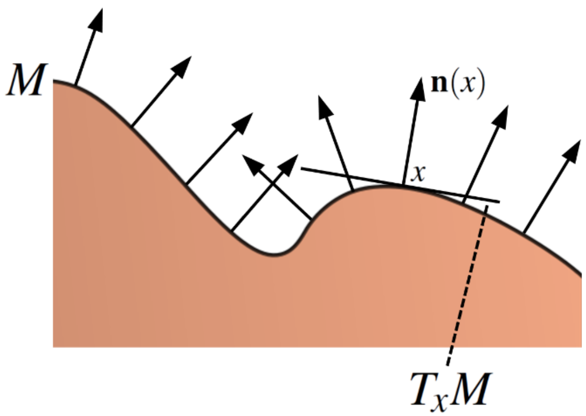
degenerate means determinant = 0 (at least one eigenvalue = 0);
bi-linear is simple: a = 0 means degenerated to
linear anyway: no critical point at all! (except constant function)
(but with more than one cell: can have max or min at vertices)

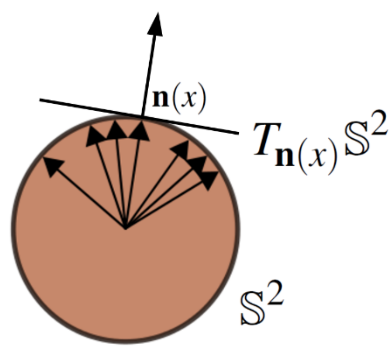


Interlude: Curvature and Shape Operator

Gauss map

$$\mathbf{n}: M \to \mathbb{S}^2$$
$$x \mapsto \mathbf{n}(x)$$





Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**

$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_x M$$

Differential of Gauss map

$$d\mathbf{n}: TM \to T\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

$$(d\mathbf{n})_x: T_x M \to T_{\mathbf{n}(x)}\mathbb{S}^2$$
$$\mathbf{v} \mapsto d\mathbf{n}(\mathbf{v})$$

Shape operator (Weingarten map)

$$\mathbf{S}: TM \to TM$$

$$\mathbf{S}_x: T_x M \to T_x M$$
$$\mathbf{v} \mapsto \mathbf{S}_x(\mathbf{v}) = d\mathbf{n}(\mathbf{v})$$

Interlude: Curvature and Shape Operator

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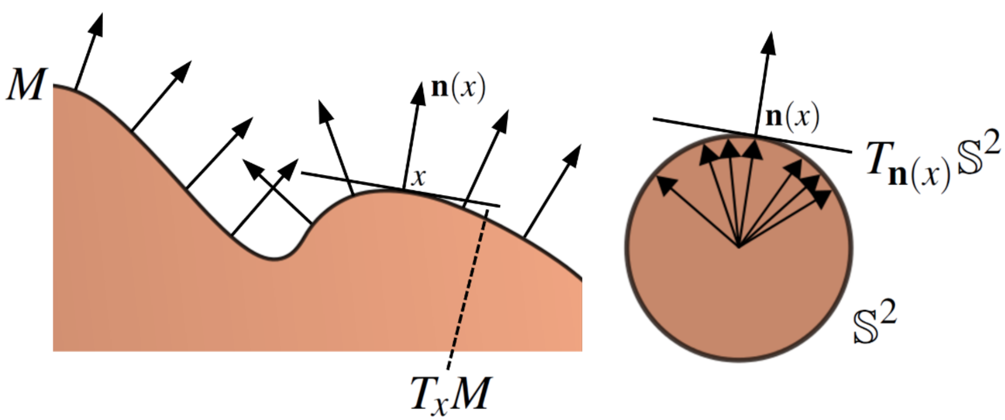

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Interlude: Curvature and Shape Operator

Gauss map

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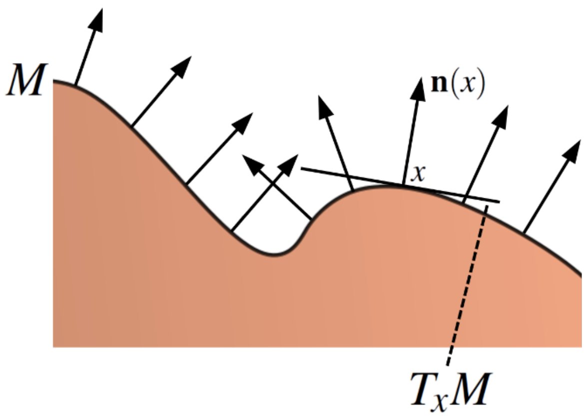
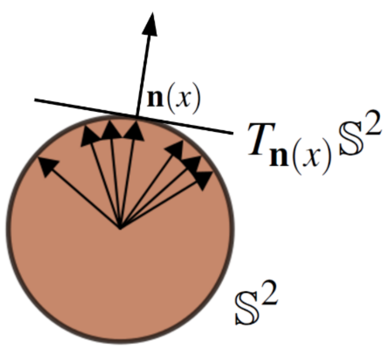




Shape operator (Weingarten map)

$$\mathbf{S}: TM \to TM$$

$$\mathbf{S}_x: T_xM \to T_xM$$
$$\mathbf{v} \mapsto \mathbf{S}_x(\mathbf{v}) = -\nabla_{\mathbf{v}}\mathbf{n}$$

(sign is convention)

Principal curvature magnitudes and directions are eigenvalues and eigenvectors of shape operator **S**

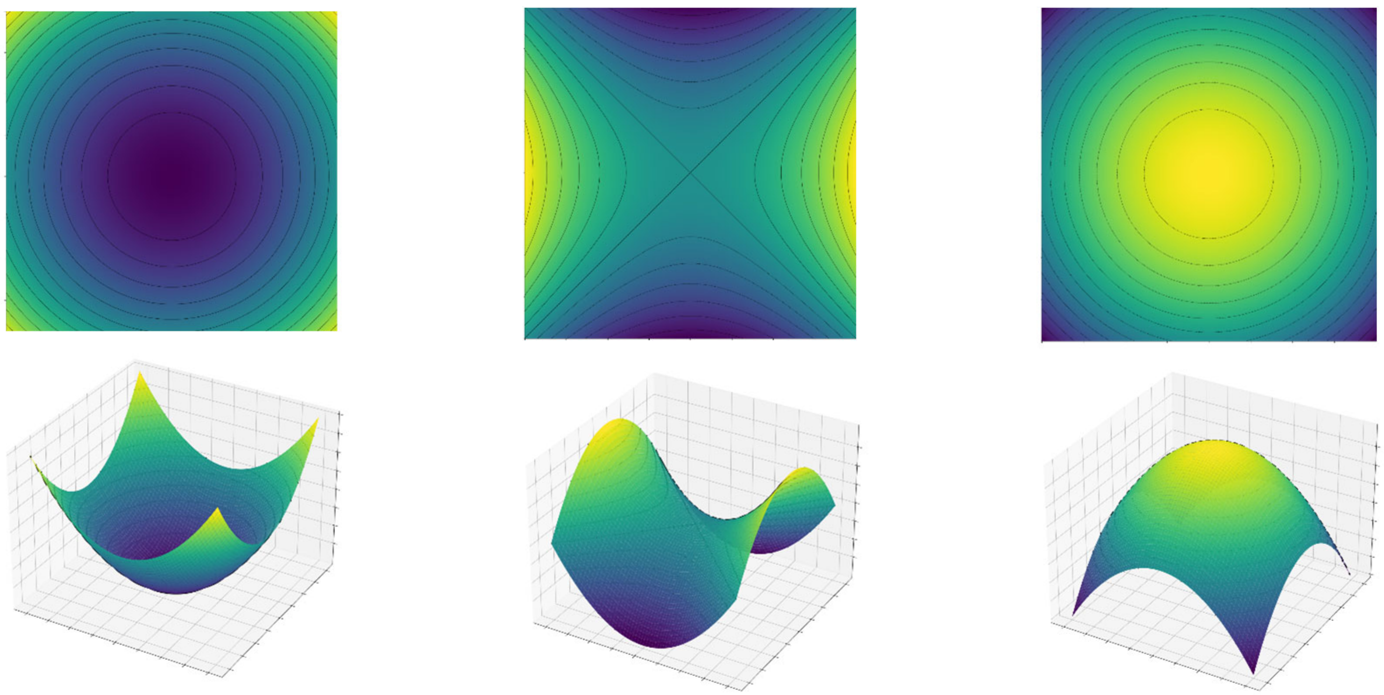
$$T_{\mathbf{n}(x)}\mathbb{S}^2 \cong T_xM$$

General Case (2D Scalar Fields)

In 2D scalar fields, only *three types* of (isolated, non-degenerate) critical points

Index of critical point: dimension of eigenspace with negative-definite Hessian

minimum (index 0) | saddle point (index 1) | maximum (index 2)



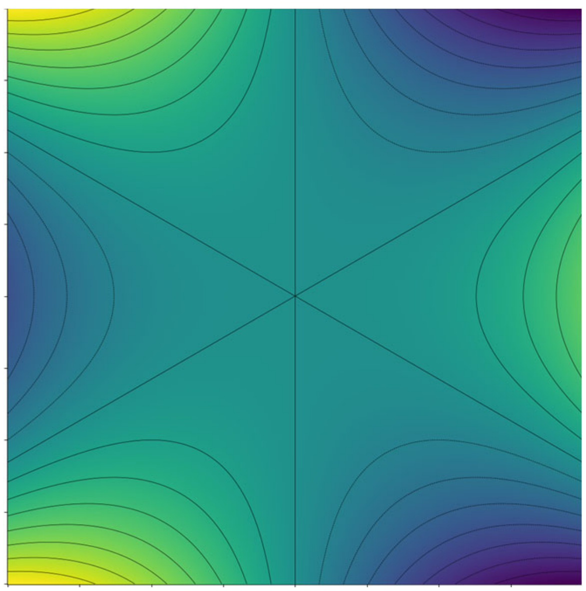
Interesting Degenerate Critical Points?

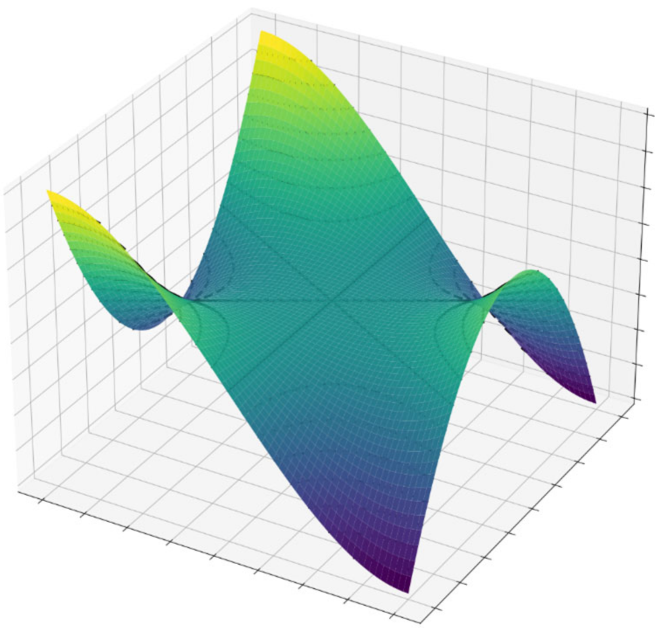
Hessian matrix is singular (determinant = 0)

- Cannot say what happens: need higher-order derivatives, …

Interesting example: monkey saddle $z = x^3 - 3xy^2$ ('third-order saddle')

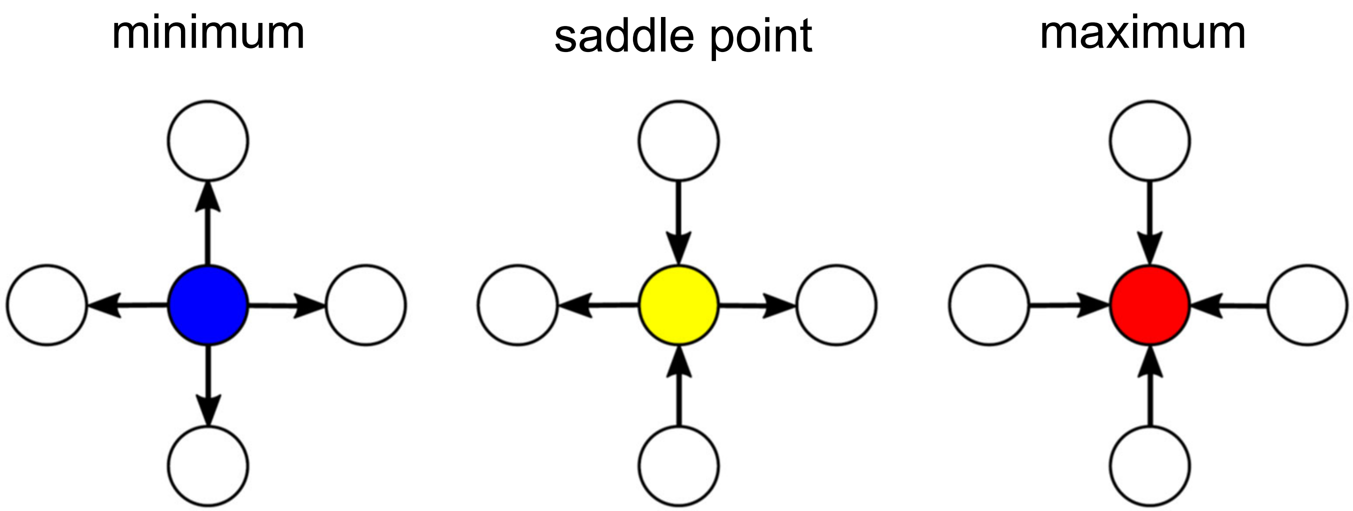
- Point (0,0) in center: Hessian = 0; Gaussian curvature = 0 (umbilical point)





Discrete Classification of Critical Points

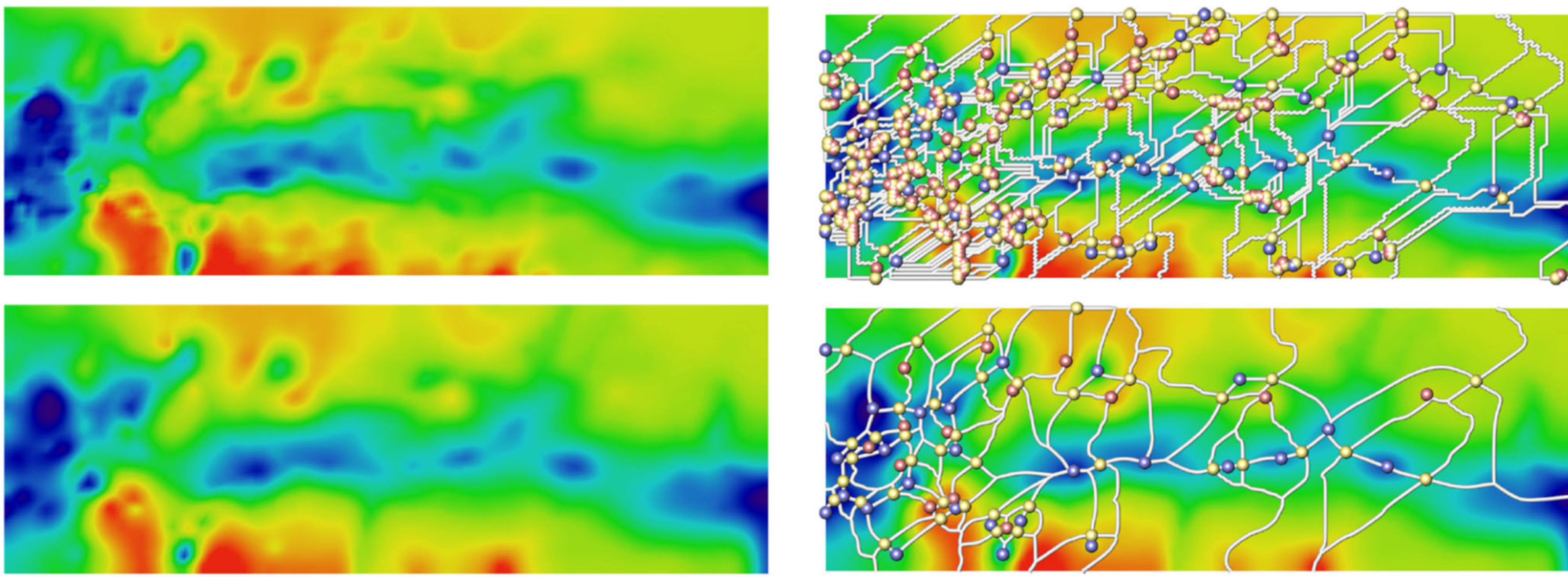
Combinatorial classification (looking at and comparing neighbors)
instead of looking at derivatives
(i.e., derivatives of the smooth function that is not known)



…toward scalar field topology, discrete Morse theory, Morse-Smale complex, …

Example: Scalar Field Simplification

Topology-based smoothing of 2D scalar fields, Weinkauf et al., 2010



Example: Differential Topology

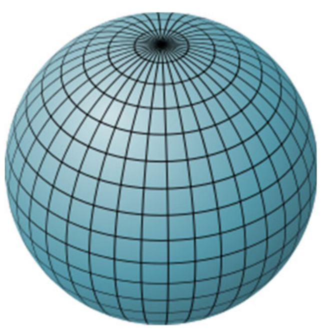
Morse theory

- Morse function: scalar function where all critical points are non-degenerate and have different critical value

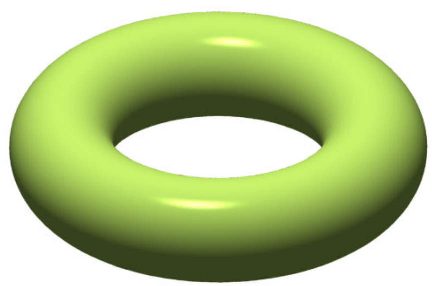
Topological invariant: Euler characteristic $\chi(M)$ of manifold M

(for 2-manifold mesh: $\chi(M) = V - E + F$)

$\chi = 2 - 2g$ (orientable)



genus $g = 0$
Euler characteristic $\chi = 2$



genus $g = 1$
Euler characteristic $\chi = 0$

genus $g = 2$
Euler characteristic $\chi = -2$

Example: Differential Topology

Morse theory

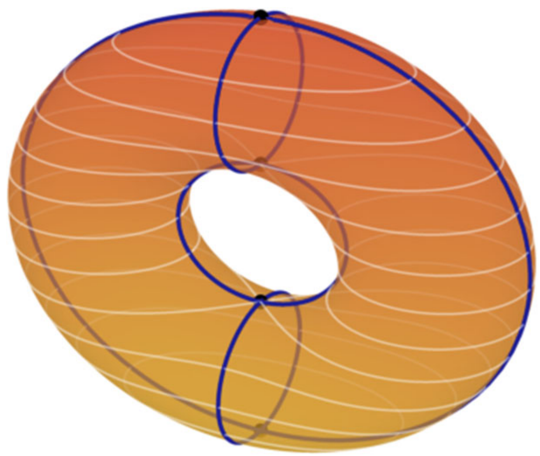
- Morse function: scalar function where all critical points are non-degenerate and have different critical value

Topological invariant: Euler characteristic $\chi(M)$ of manifold M

$$\chi(M) = \sum_{i=0}^{n} (-1)^i m_i$$

m_i: number of critical points with index i

n: dimensionality of M



genus $g(M) = 1$

Euler characteristic $\chi(M) = 0 \; (= 1 - 2 + 1)$

scalar function on torus is height function $f(x,y,z) = z$:

1 min, 1 max, 2 saddles

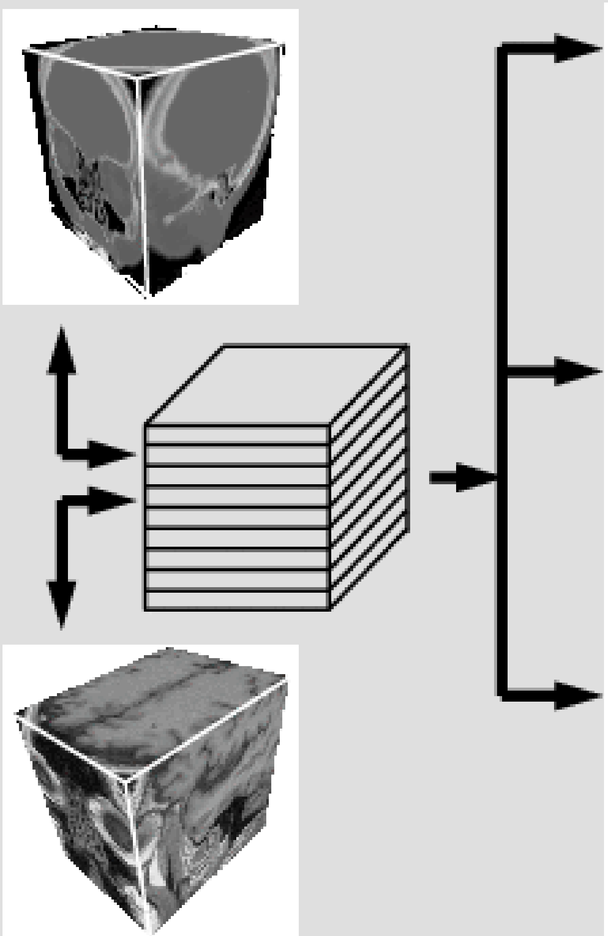
critical points are where

$$df(x,y,z) = 0$$

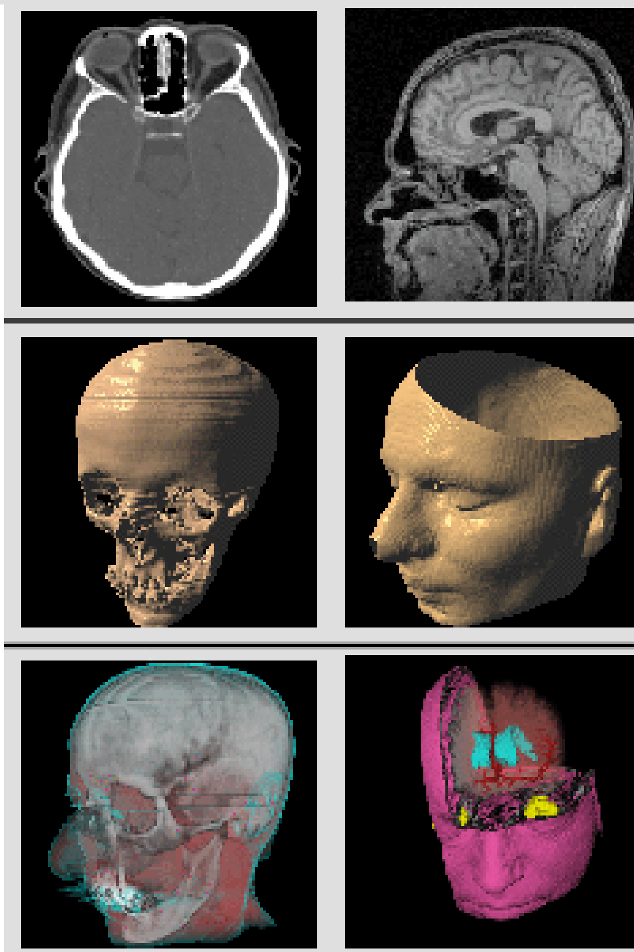
(tangent plane horizontal)

Volume Visualization

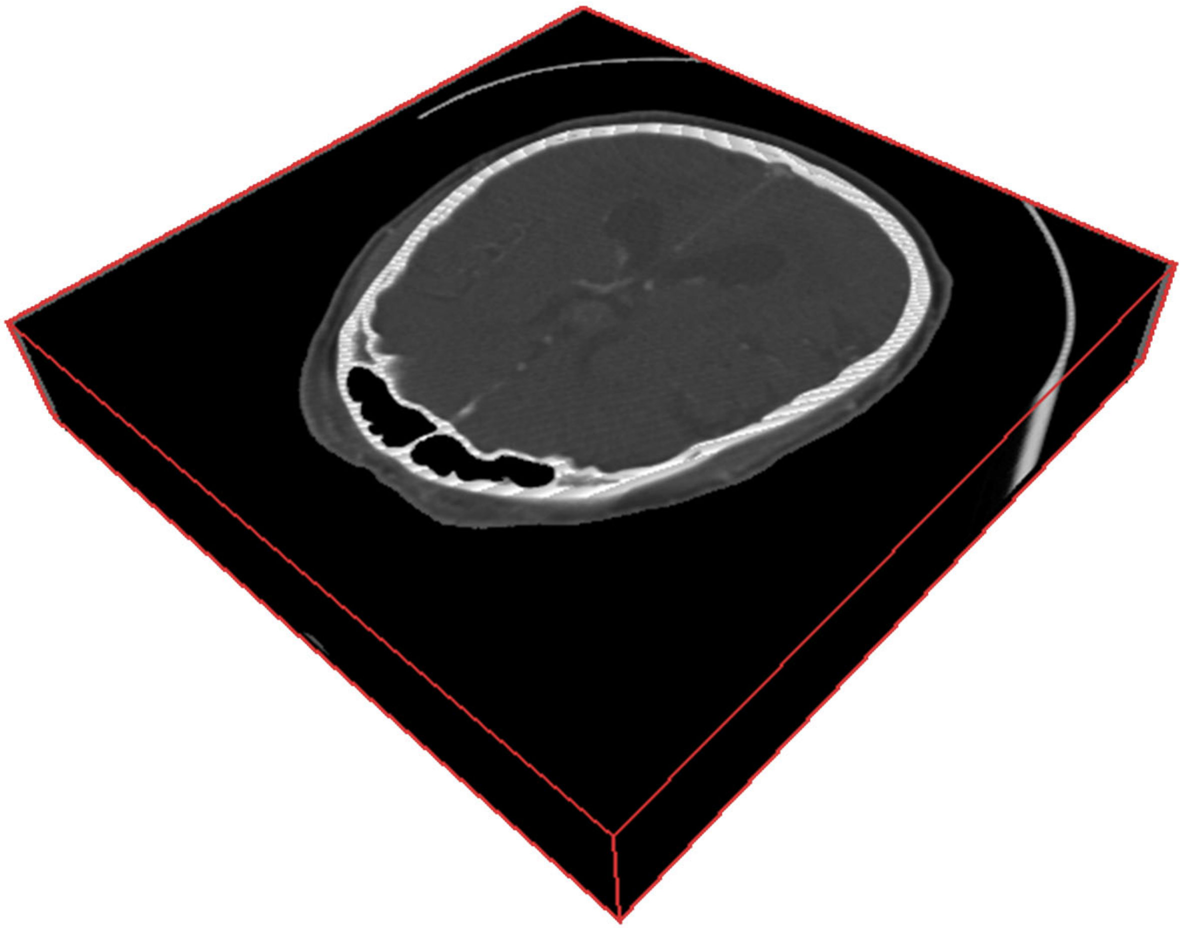
Volume Visualization



- 2D visualization
  slice images
  (or multi-planar
  reformatting MPR)
- *Indirect*
  3D visualization
  isosurfaces
  (or surface-shaded
  display: SSD)
- *Direct*
  3D visualization
  (direct volume
  rendering: DVR)



Direct Volume Rendering



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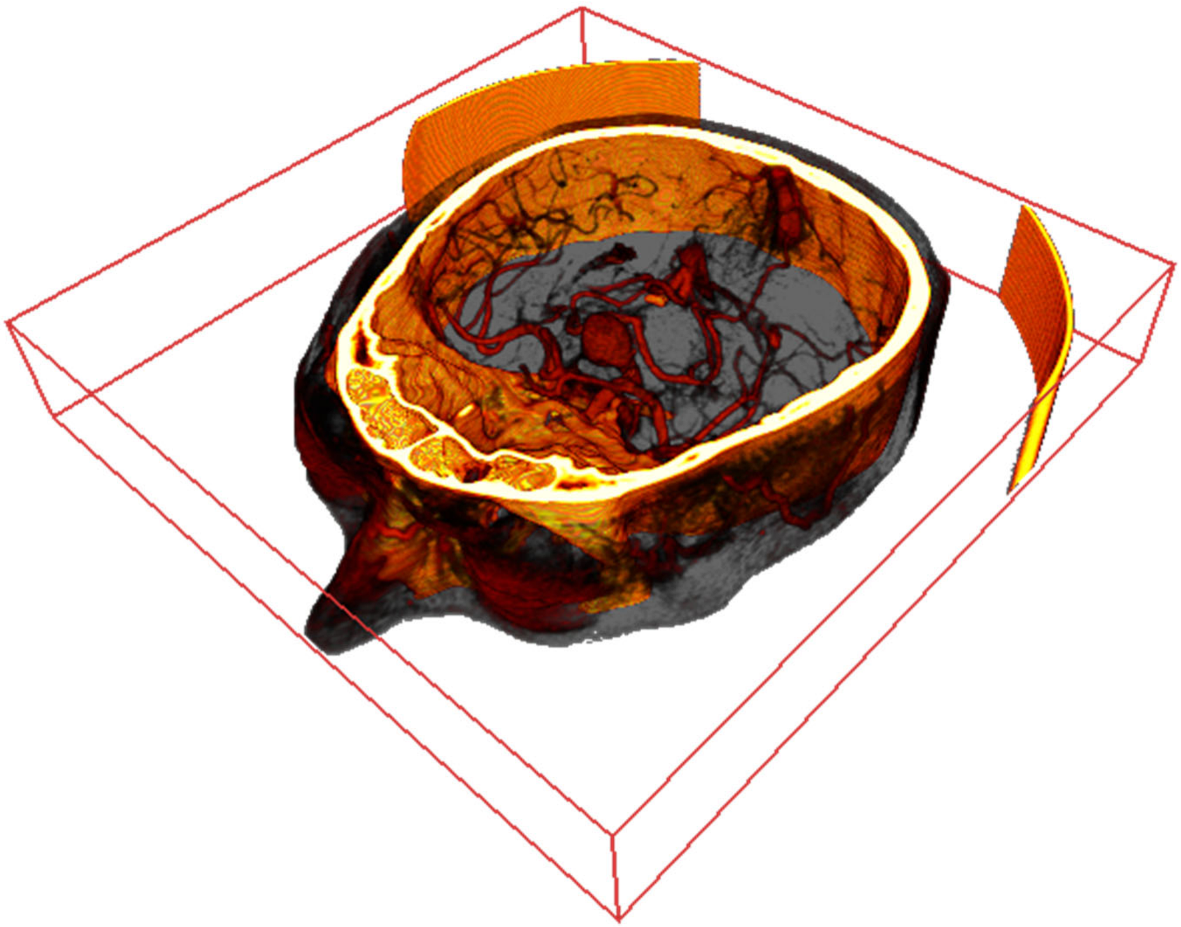
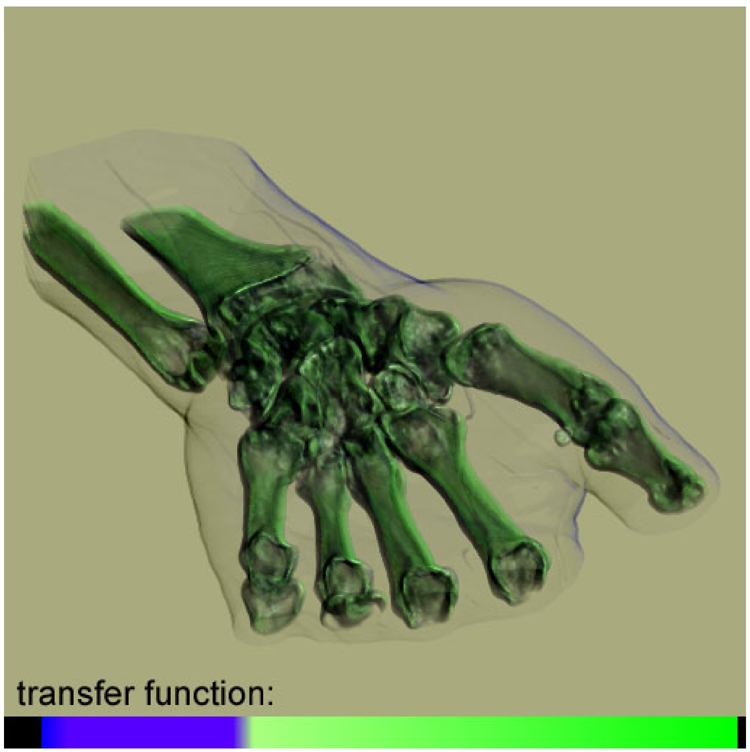
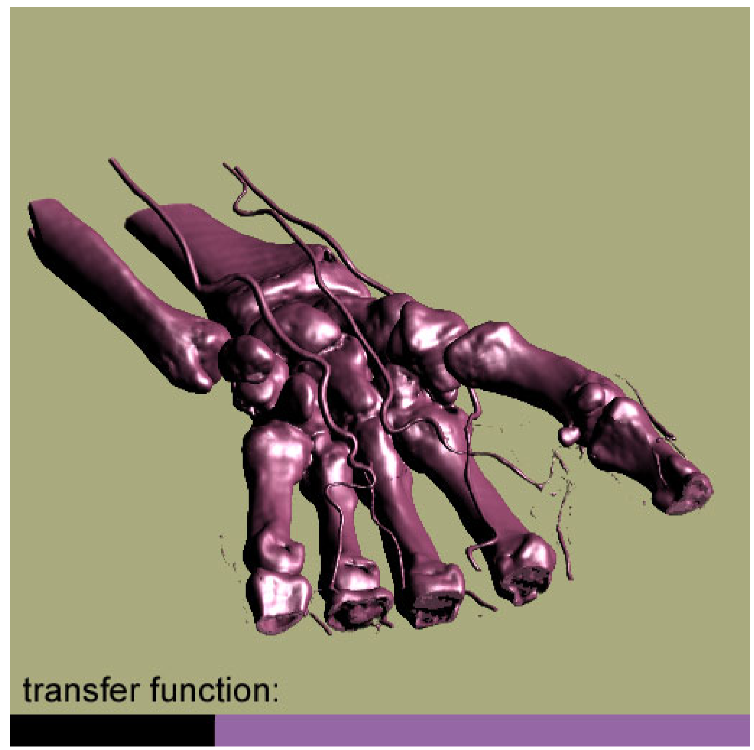


Transparent Volumes vs. Isosurfaces

The *transfer function* assigns *optical properties* to data

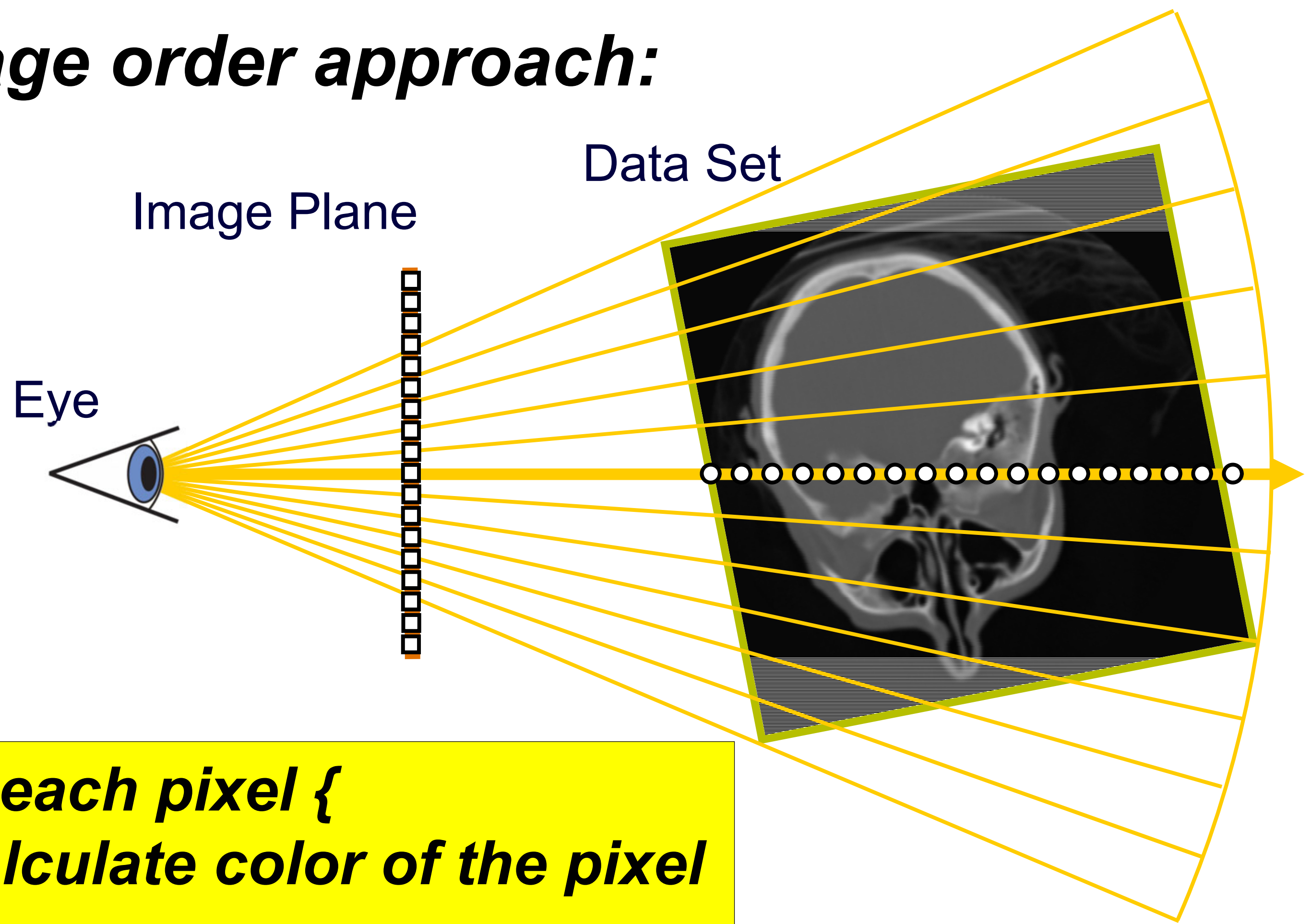
- Translucent volumes
- But also: isosurface rendering using step function as transfer function





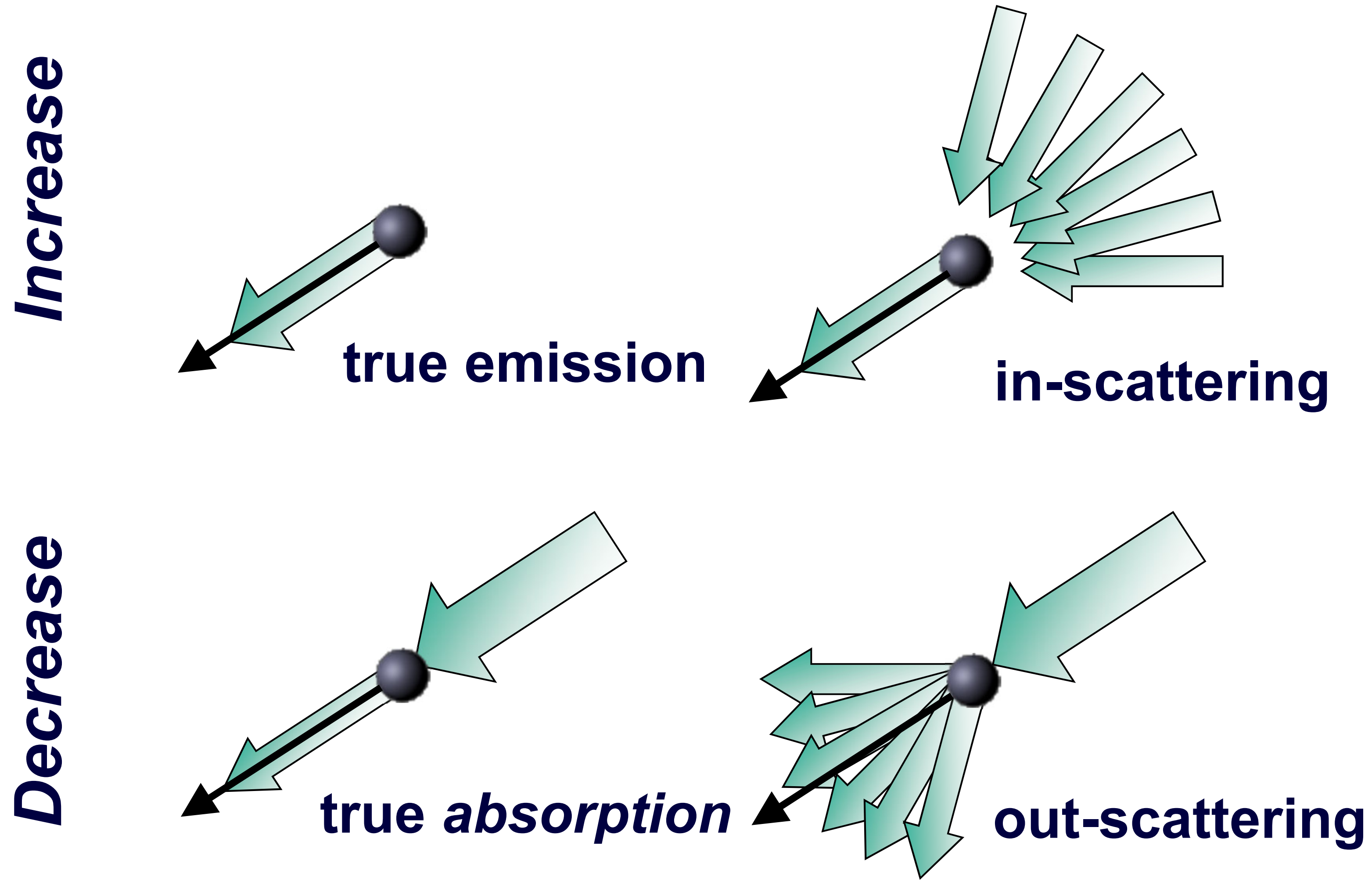
Direct Volume Rendering

Image order approach:



For each pixel {
calculate color of the pixel
}

Physical Model of Radiative Transfer



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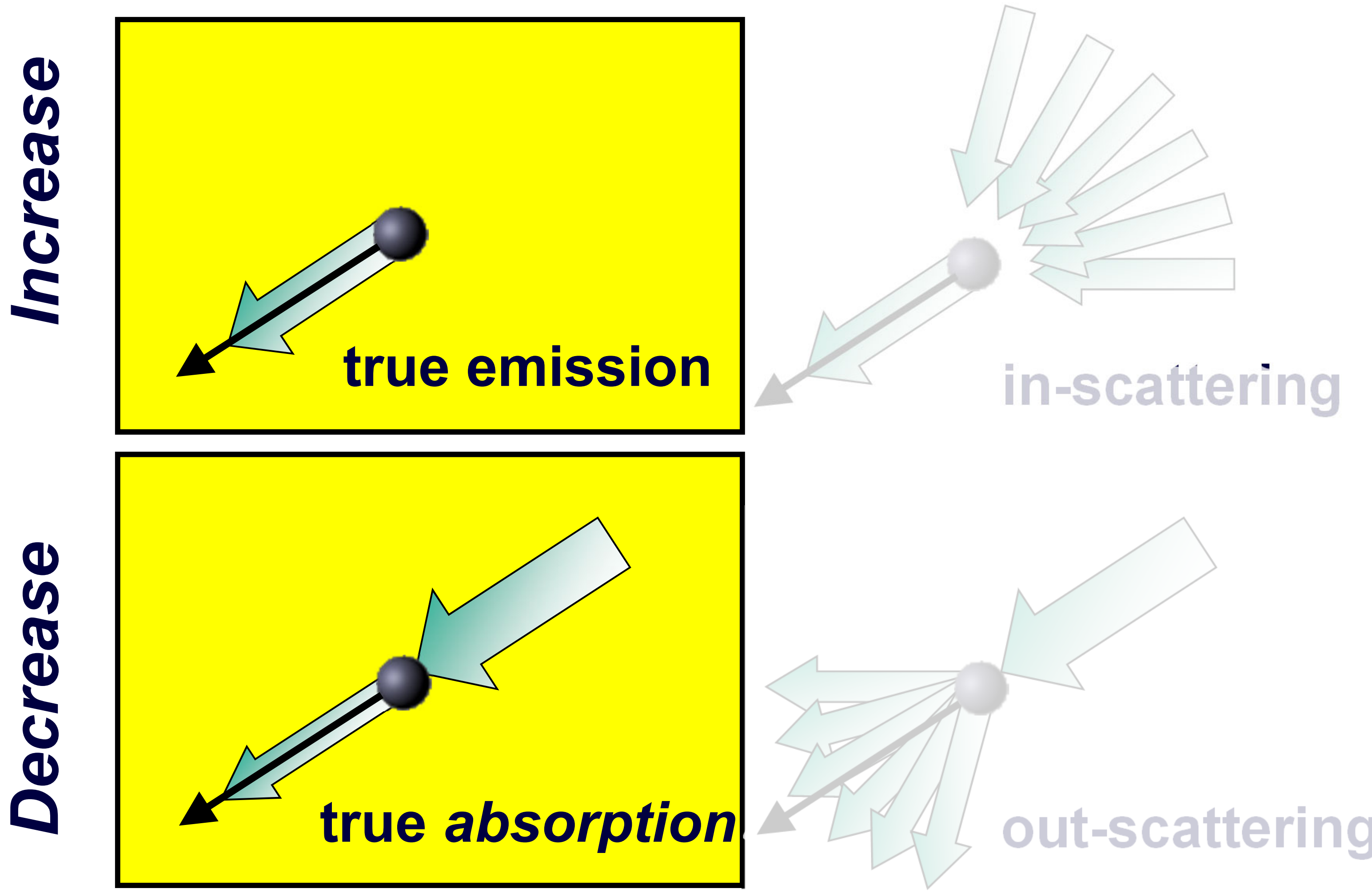
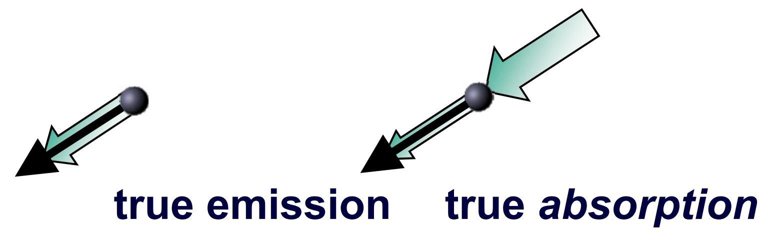


Volume Rendering Integral

Volume rendering integral
for *Emission Absorption* model



$$I(s) \;=\; I(s_0)\, e^{-\tau(s_0,s)} + \int_{s_0}^{s} q(\tilde{s})\, e^{-\tau(\tilde{s},s)}\, \mathrm{d}\tilde{s}$$

Numerical solutions:

Back-to-front compositing

$$C_i' \;=\; C_i + (1 - A_i)C_{i-1}'$$

Front-to-back compositing

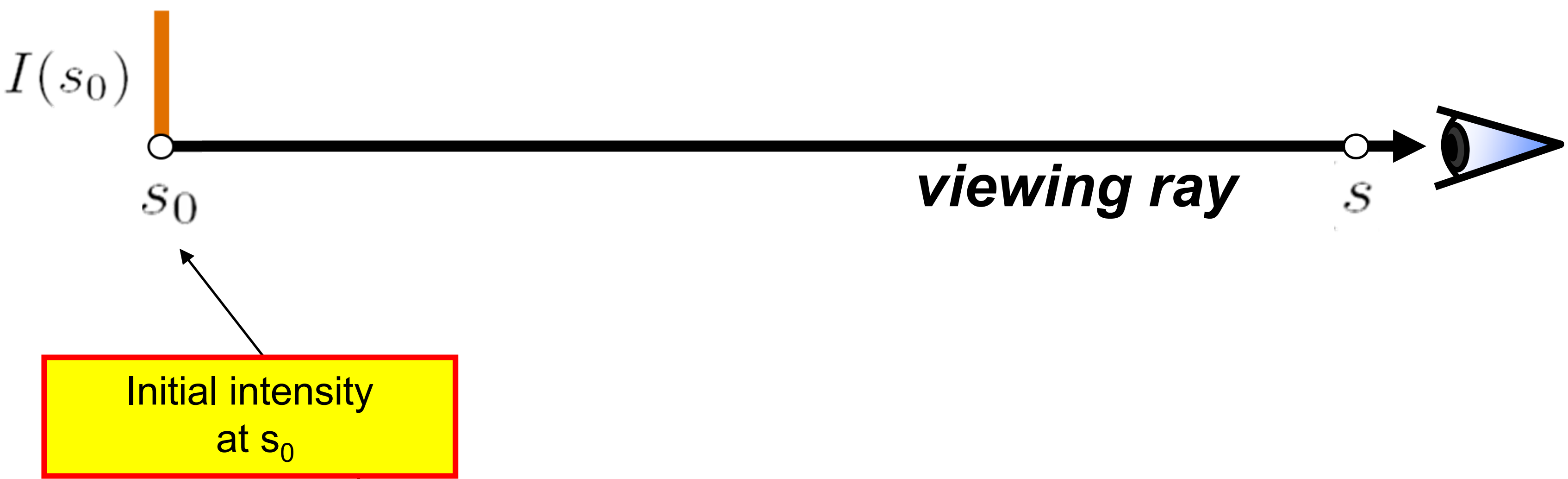
$$C_i' \;=\; C_{i+1}' + (1 - A_{i+1}')C_i$$

$$A_i' \;=\; A_{i+1}' + (1 - A_{i+1}')A_i$$

Volume Rendering Integral

How do we determine the radiant energy along the ray?

Physical model: emission and absorption, no scattering



$$I(s) = I(s_0)$$

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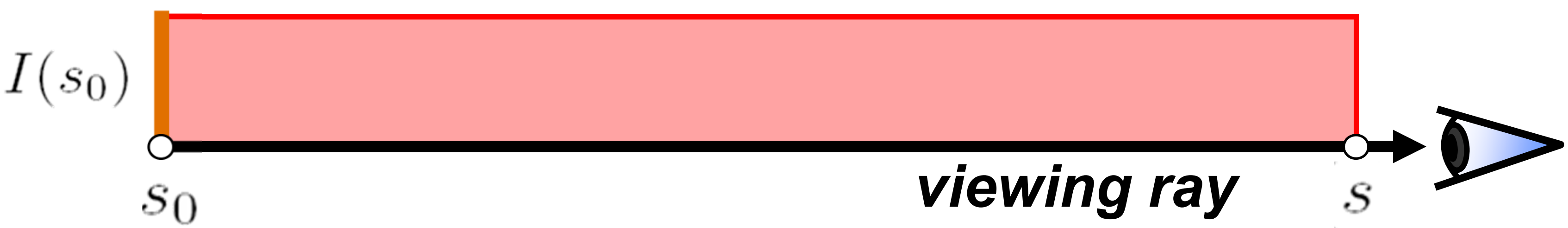

Initial intensity at s_0

$$I(s) = \boxed{I(s_0)}$$

Without absorption all the initial radiant energy would reach the point ***s.***

Volume Rendering Integral

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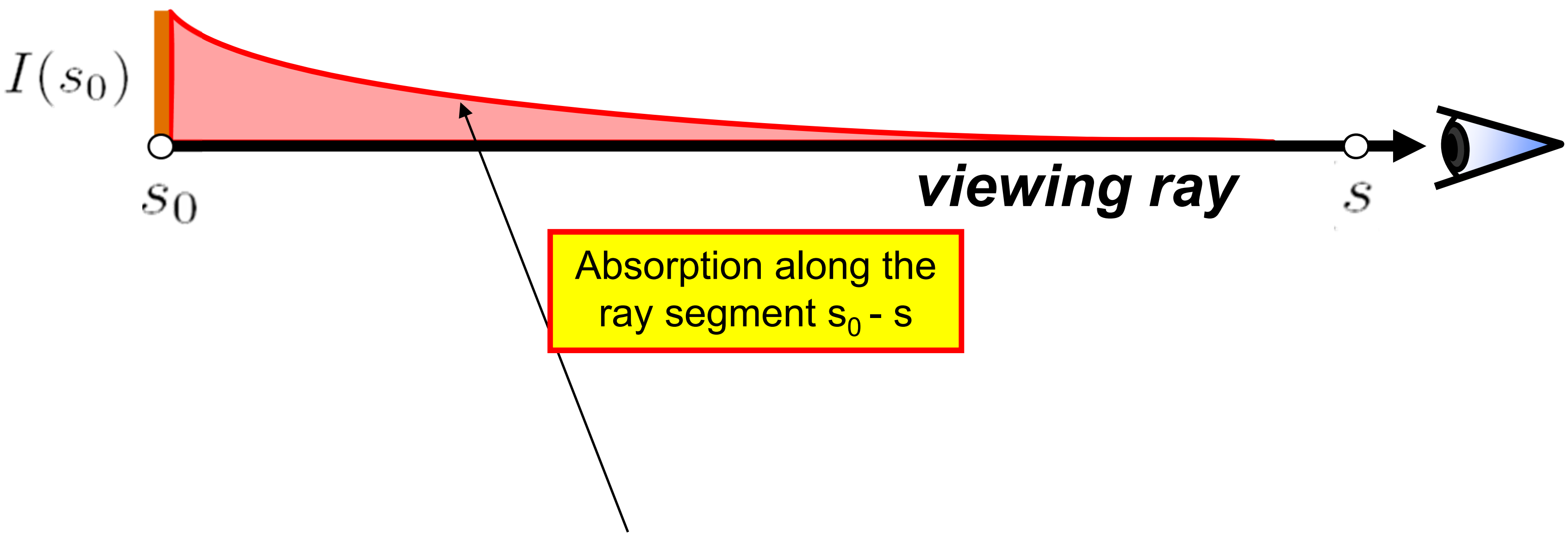

$$I(s) = I(s_0) e^{-\tau(s_0, s)}$$

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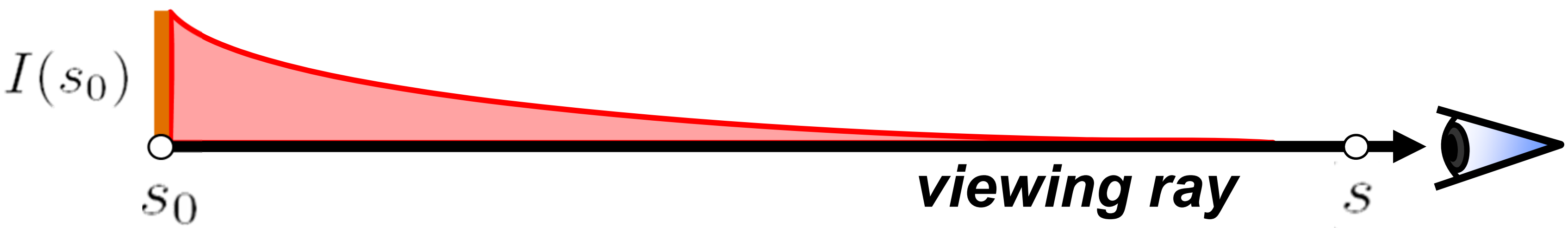

$$I(s) = I(s_0)\, e^{-\tau(s_0, s)}$$

Optical depth τ

Absorption κ

$$\tau(s_1, s_2) = \int_{s_1}^{s_2} \kappa(s)\, ds.$$

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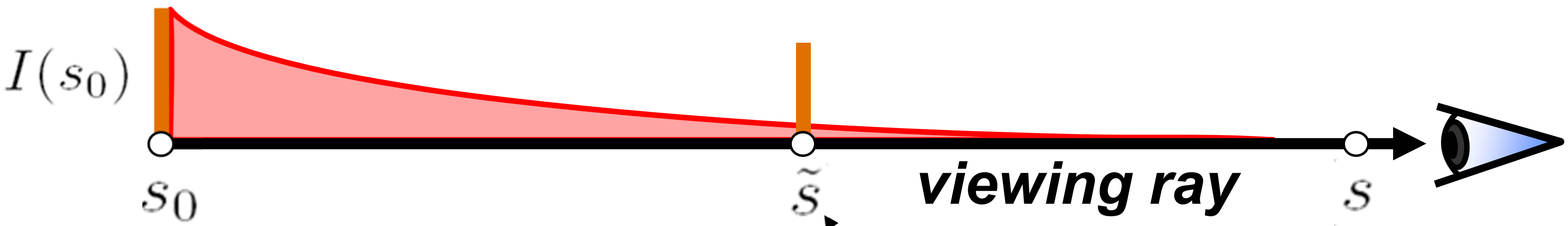

One point $\tilde{s}$ along the viewing ray emits additional radiant energy.

Active emission at point $\tilde{s}$

$$I(s) = I(s_0)\, e^{-\tau(s_0,s)} + q(\tilde{s})$$

Volume Rendering Integral

How do we determine the radiant energy along the ray?

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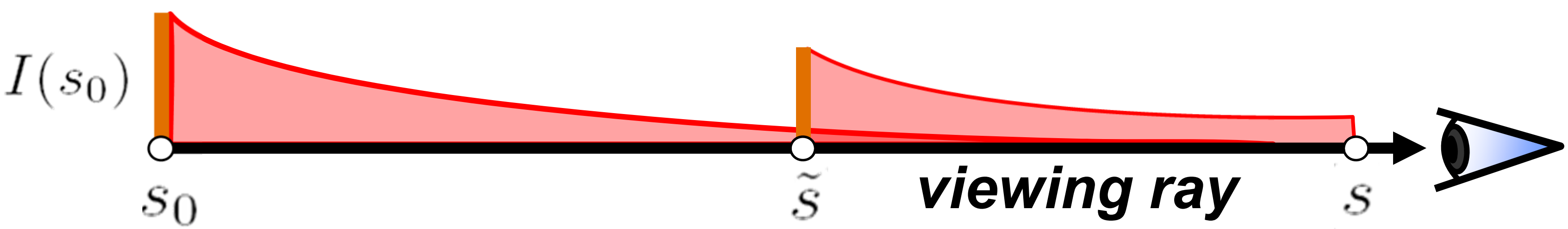

Every point $\tilde{s}$ along the viewing ray emits additional radiant energy

$$I(s) = I(s_0)\, e^{-\tau(s_0,s)} + \int_{s_0}^{s} q(\tilde{s})\, e^{-\tau(\tilde{s},s)}\, d\tilde{s}$$

Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama