King Abdullah University of Science and Technology KAUST

# CS 247 – Scientific Visualization
# Lecture 8: Scalar Fields, Pt. 4

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# Reading Assignment #4 (until Feb 21)

## Read (required):

- Real-Time Volume Graphics book, Chapter 5 until 5.4 inclusive (*Terminology, Types of Light Sources, Gradient-Based Illumination, Local Illumination Models*)
- Paper:
  *Marching Cubes: A high resolution 3D surface construction algorithm*,
  Bill Lorensen and Harvey Cline, ACM SIGGRAPH 1987
  [> 17,700 citations and counting...]

  **`https://dl.acm.org/doi/10.1145/37402.37422`**

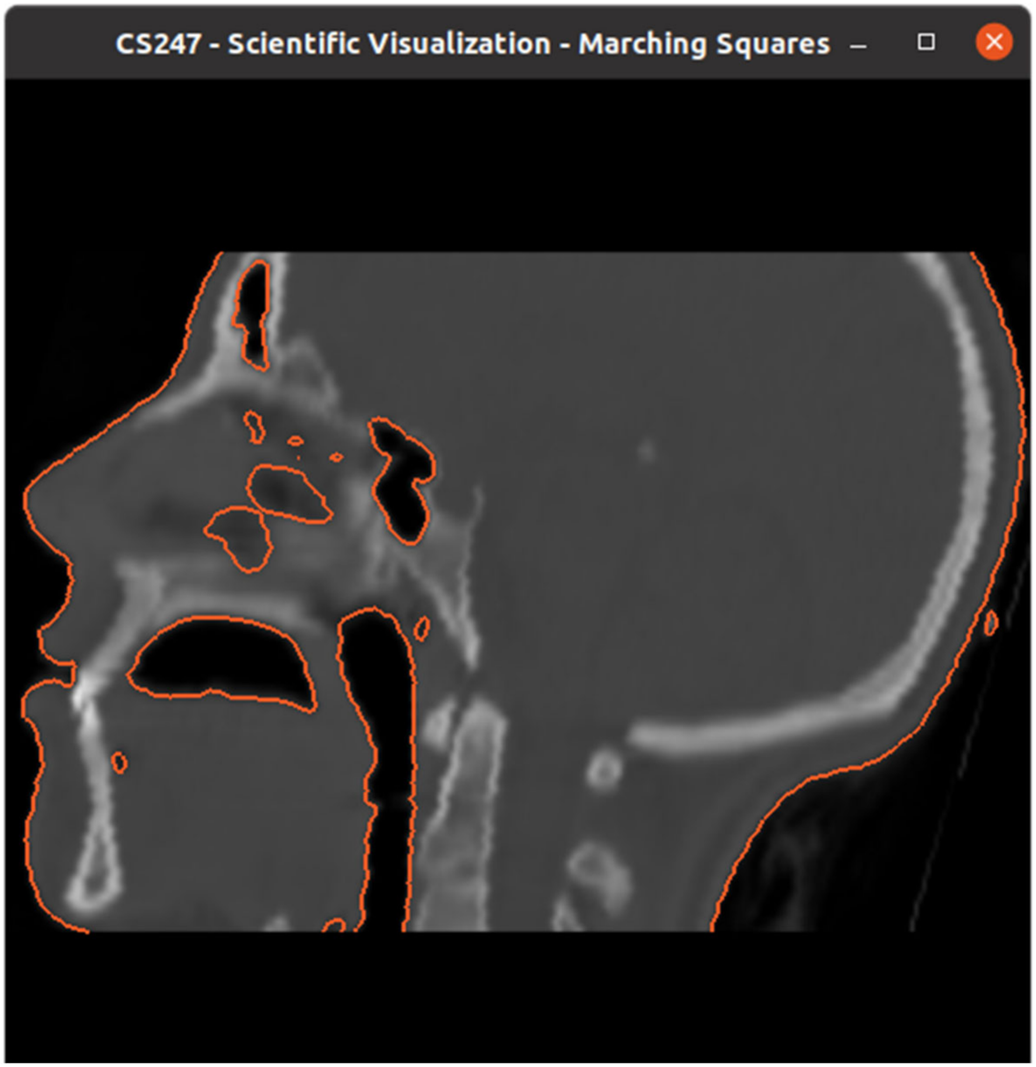
## Read (optional):

- Paper:
  *Flying Edges,* William Schroeder et al., IEEE LDAV 2015

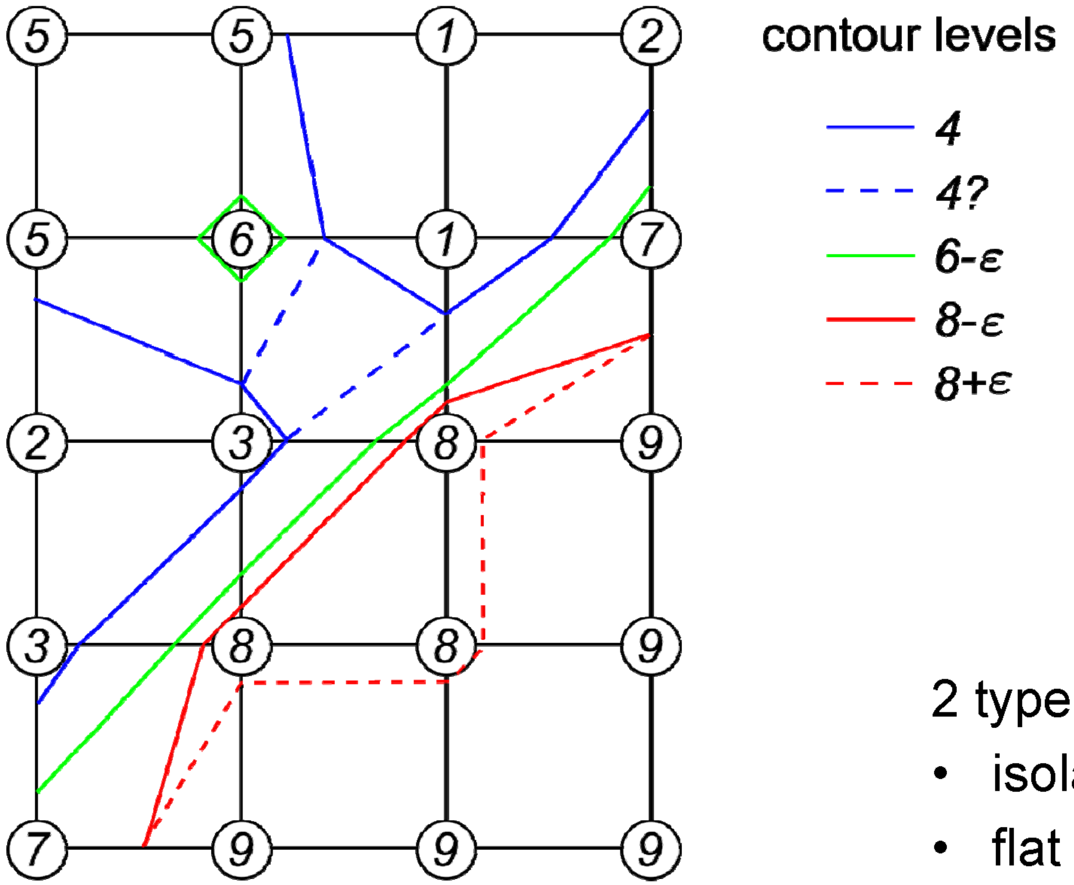
  **`https://ieeexplore.ieee.org/document/7348069`**

# Scalar Fields

# Marching Squares Example



# Marching Squares Example




2 types of degeneracies:

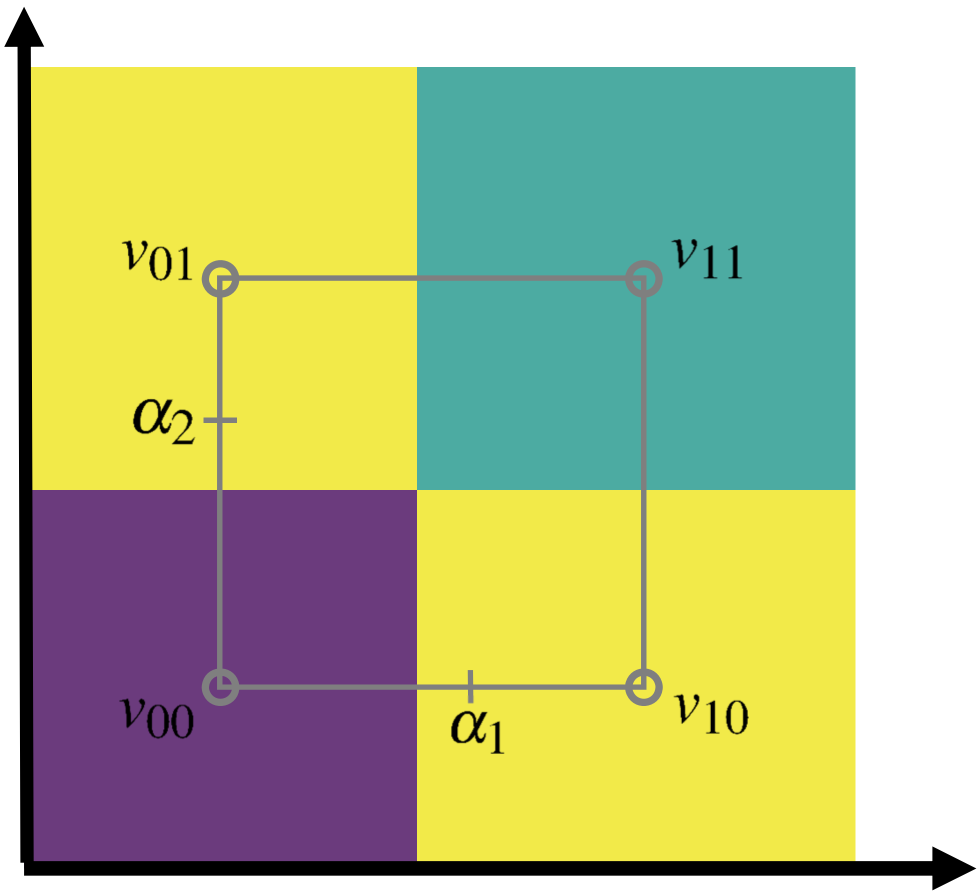
- isolated points (*c*=6)
- flat regions (*c*=8)

# Sample Locations and Interpolation

Consider area between 2x2 adjacent samples (e.g., pixel centers):

Given any (fractional) position

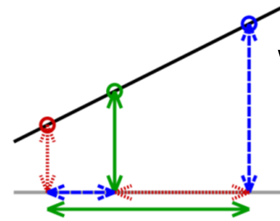
$$\alpha_1 := x_1 - \lfloor x_1 \rfloor \qquad \alpha_1 \in [0.0, 1.0)$$
$$\alpha_2 := x_2 - \lfloor x_2 \rfloor \qquad \alpha_2 \in [0.0, 1.0)$$



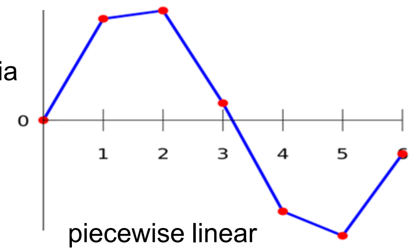
# Linear Interpolation / Convex Combinations

Linear interpolation in 1D:

$$f(\alpha) = (1-\alpha)v_1 + \alpha v_2$$



wikipedia




Line embedded in 2D (linear interpolation of vertex coordinates/attributes):

$$f(\alpha_1, \alpha_2) = \alpha_1 v_1 + \alpha_2 v_2 \qquad \alpha_1 + \alpha_2 = 1$$

$$f(\alpha) = v_1 + \alpha(v_2 - v_1) \qquad \alpha = \alpha_2$$

Line segment: $\alpha_1, \alpha_2 \geq 0$ ($\rightarrow$ convex combination)

Compare to line parameterization with parameter t:

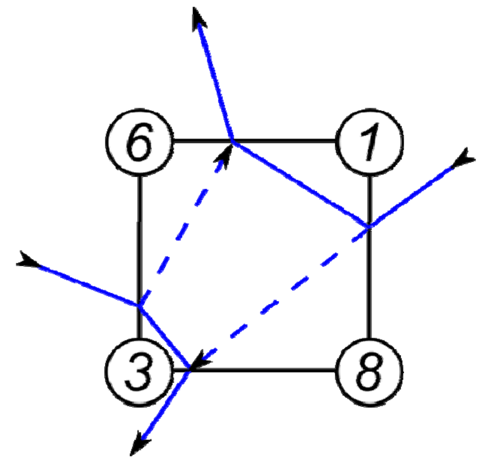
$$v(t) = v_1 + t(v_2 - v_1)$$

# *Ambiguities of contours*

What is the correct contour of $c$=4?

Two possibilities, both are orientable:

- connect high values ———————
- connect low values - - - - - - - - - -




Answer: correctness depends on interior values of $f(x)$.

But: different interpolation schemes are possible.

Better question: What is the correct contour with respect to bilinear interpolation?

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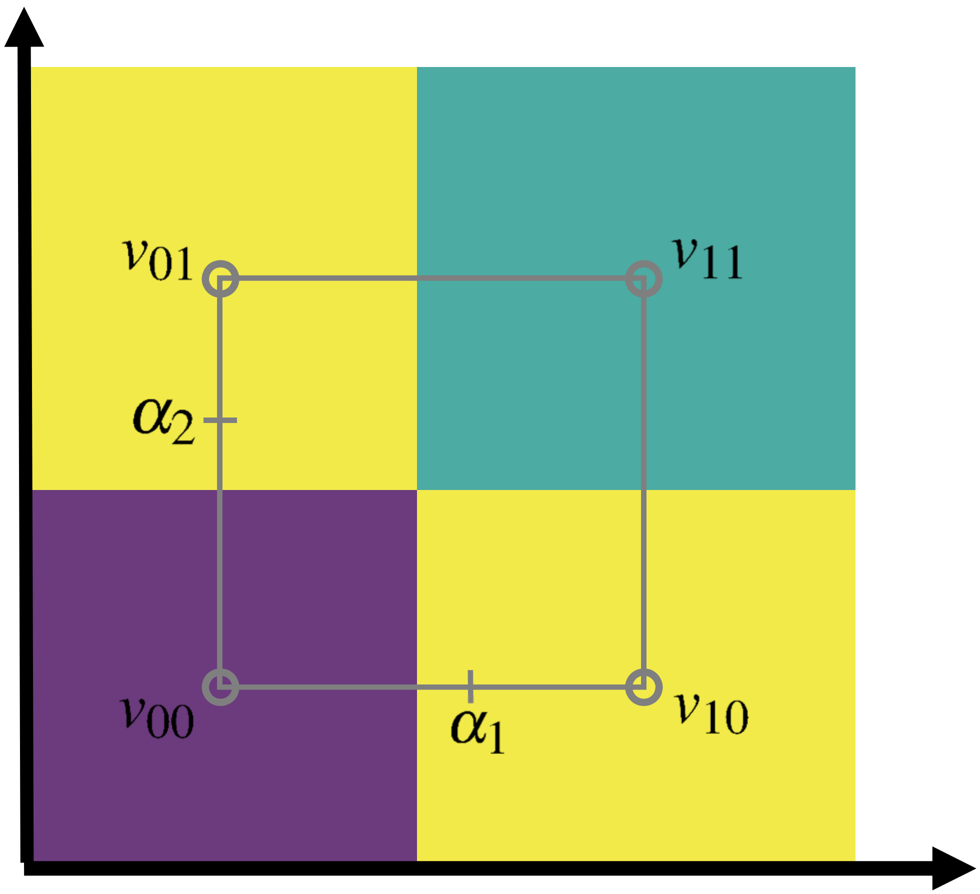

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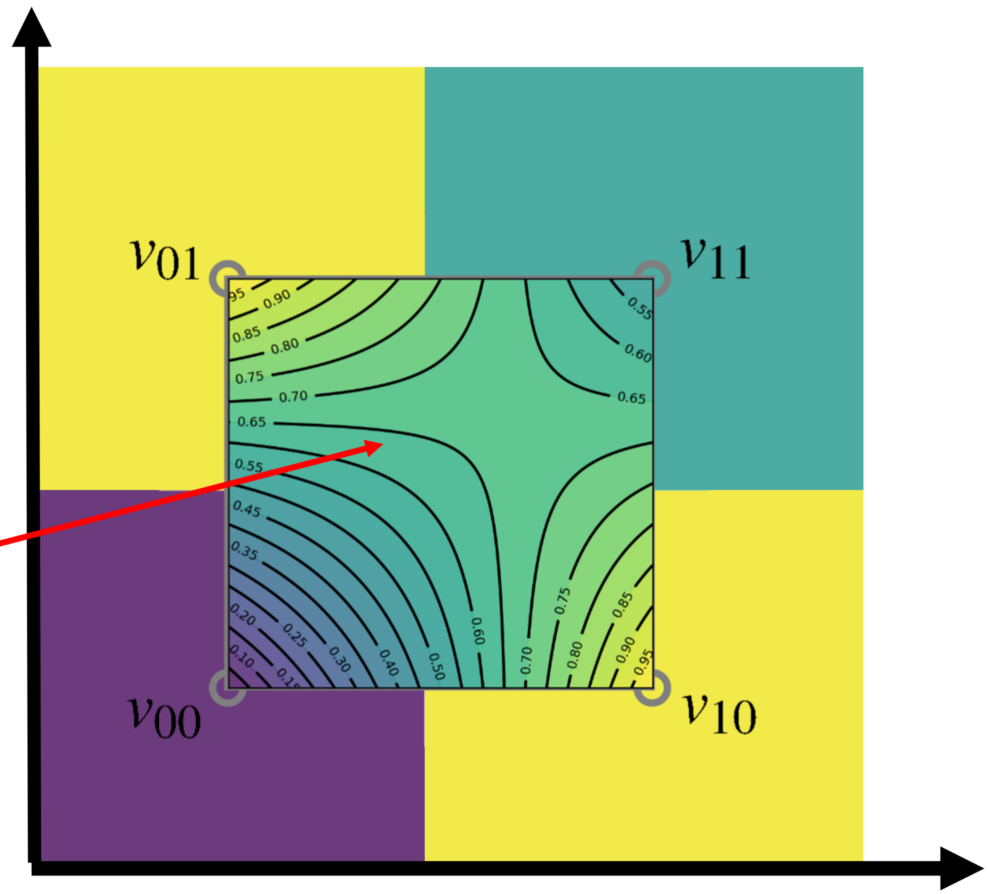
bilinear interpolation
(not marching squares!)



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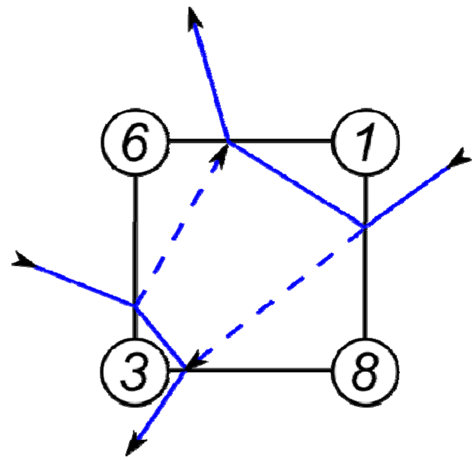



Answer: correctness depends on interior values of $f(x)$.

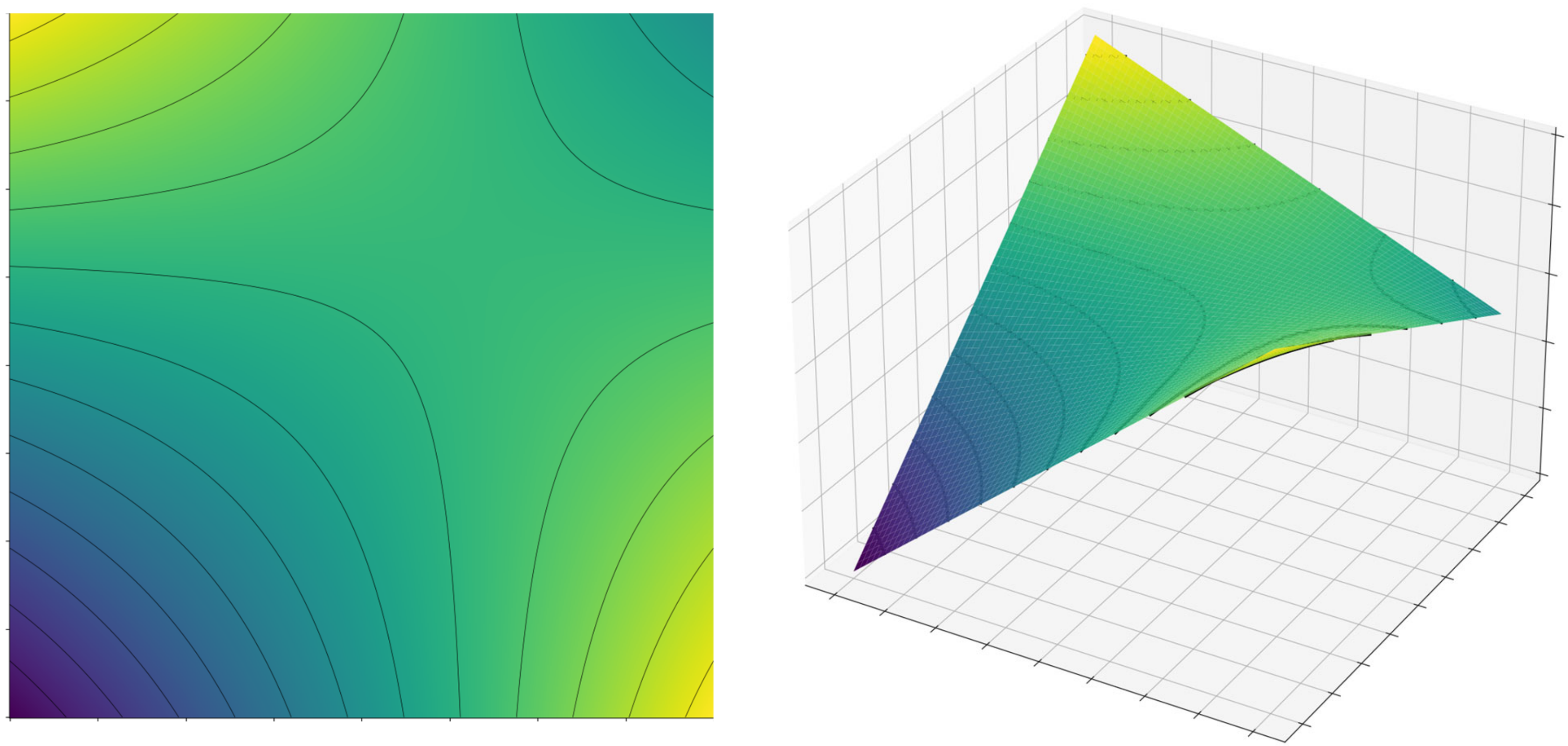
But: different interpolation schemes are possible.

Better question: What is the correct contour with respect to bilinear interpolation?

# Bi-Linear Interpolation

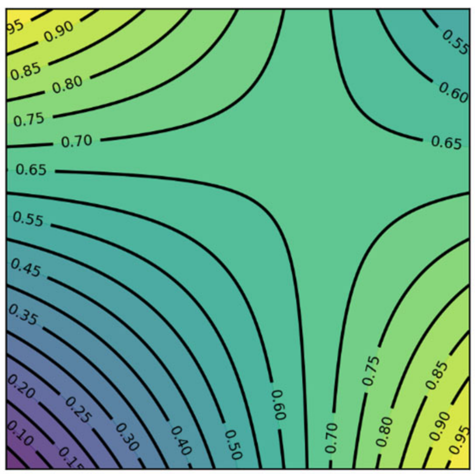
Consider area between 2x2 adjacent samples

Example: 1.0 at top-left and bottom-right, 0.0 at bottom-left, 0.5 at top-right

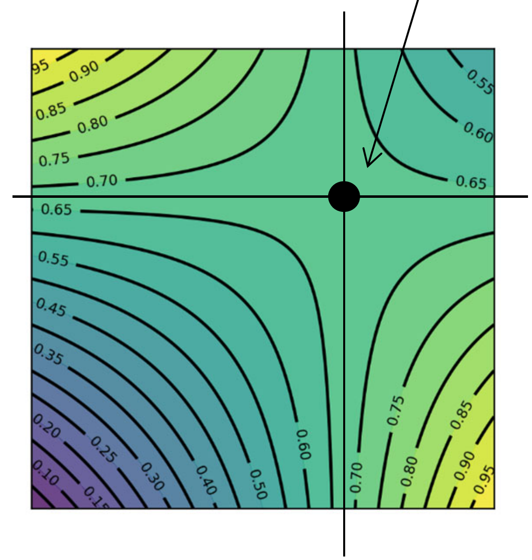


# Bi-Linear Interpolation: Critical Points

Critical points are where the gradient vanishes (i.e., is the zero vector)




critical point
(saddle point)




here, the critical
value is 2/3=0.666…

“Asymptotic decider”: resolve ambiguous configurations (6 and 9) by comparing specific iso-value with critical value (scalar value at critical point)

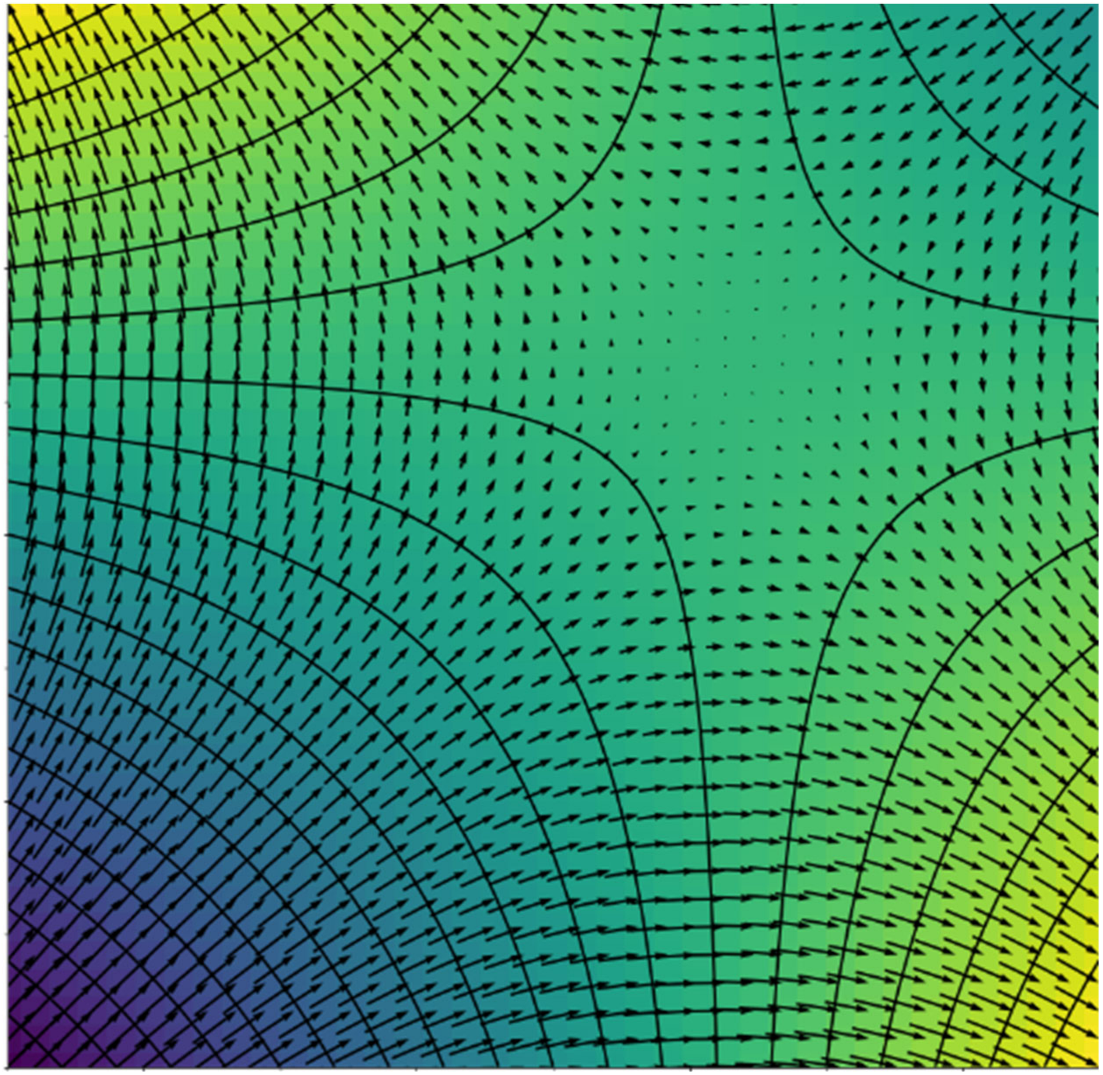
# Bi-Linear Interpolation: Critical Points

Compute gradient

Note that isolines are
farther apart where
gradient is smaller

Note the horizontal and
vertical lines where
gradient becomes
vertical/horizontal

Note the critical point



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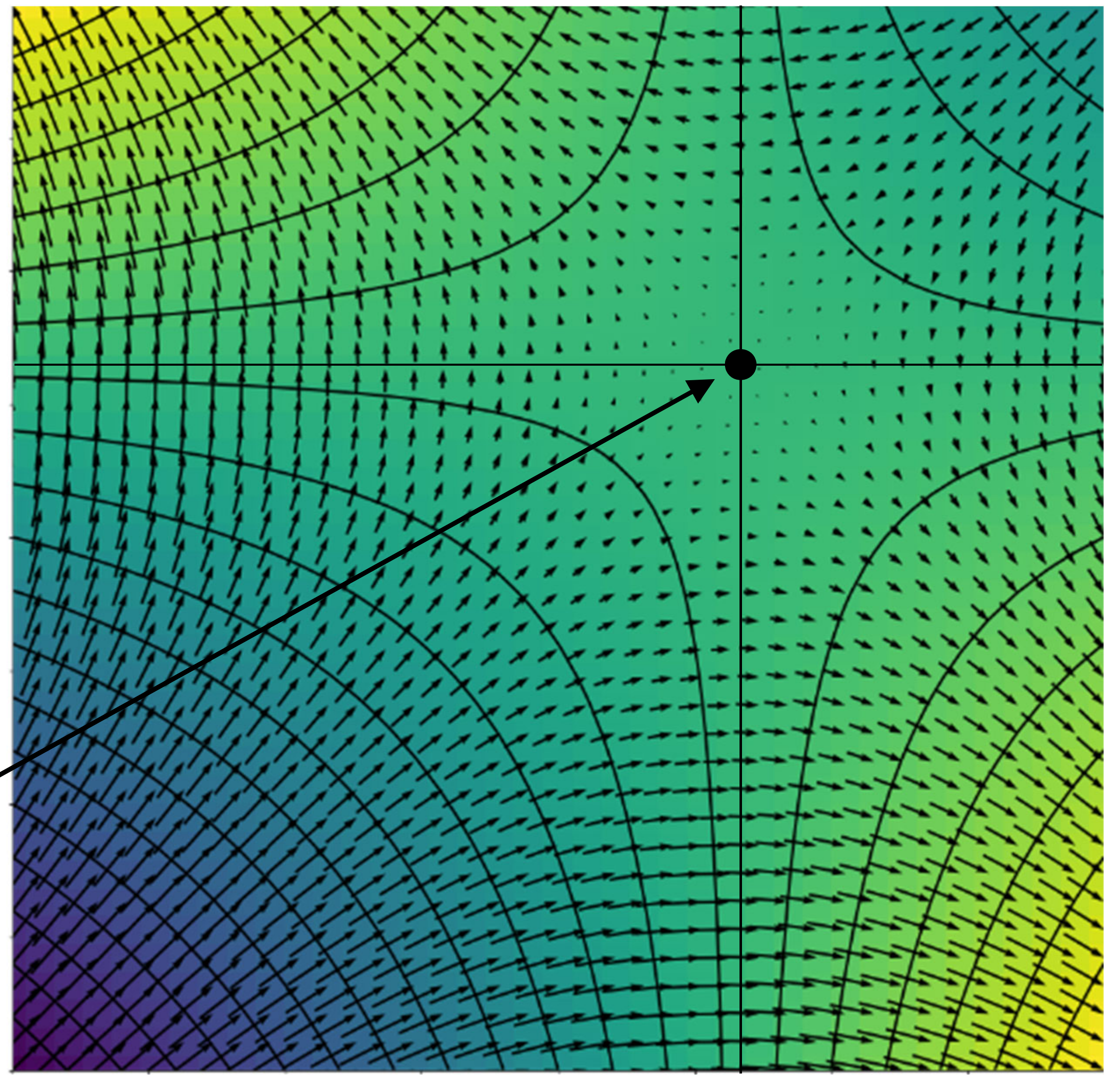


# Interlude: Implicit Function Theorem

When can I write an implicit function in $\mathbb{R}^{n+m}$ such that it is the graph of a function $f: \mathbb{R}^n \to \mathbb{R}^m$ *at least locally?*

That is: is this implicitly described function an *n*-manifold embedded in $\mathbb{R}^{n+m}$ ? (with local coordinates in $\mathbb{R}^n$ )

$$G(f) := \{(x, f(x)) | x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^m \simeq \mathbb{R}^{n+m}$$

Theorem: if *m* x *m* Jacobian matrix is invertible

(easier for scalar field: check if gradient of $f$ is non-zero)

See **https://en.wikipedia.org/wiki/Implicit_function_theorem**

General result: *constant rank theorem*

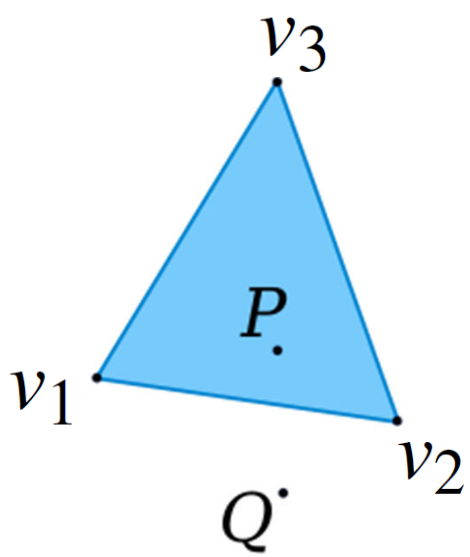
# Linear Interpolation / Convex Combinations

**Linear** combination ($n$-dim. space):

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$$

**Affine** combination: Restrict to $(n-1)$-dim. subspace:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \sum_{i=1}^{n} \alpha_i = 1$$




**Convex** combination: $\alpha_i \geq 0$

(restrict to simplex in subspace)

# Linear Interpolation / Convex Combinations

The weights $\alpha_i$ are the $n$ normalized ***barycentric*** coordinates

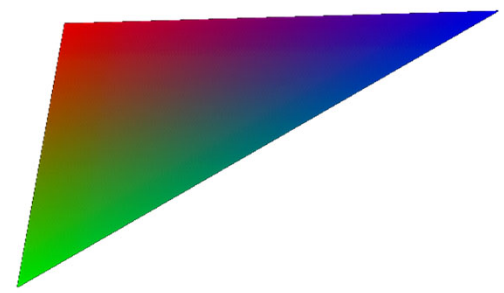
$\rightarrow$ linear attribute interpolation in simplex

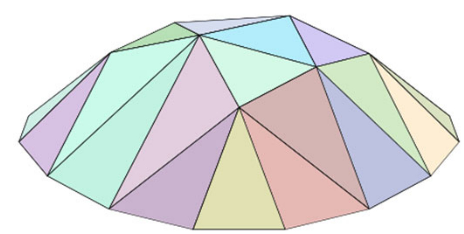
$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \sum_{i=1}^{n} \alpha_i = 1$$

$$\alpha_i \geq 0$$

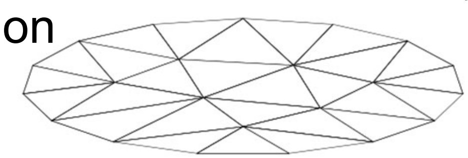
attribute interpolation





spatial position interpolation

wikipedia



# Linear Interpolation / Convex Combinations

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$$

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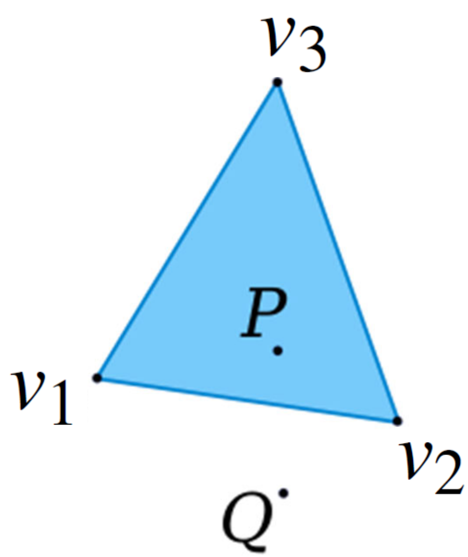
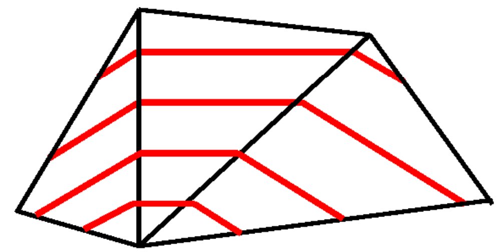



Can re-parameterize to get $(n-1)$ ***affine*** coordinates:

$$\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 &= \\ \tilde{\alpha}_1 (v_2 - v_1) + \tilde{\alpha}_2 (v_3 - v_1) + v_1 \\ \tilde{\alpha}_1 &= \alpha_2 \\ \tilde{\alpha}_2 &= \alpha_3 \end{aligned}$$

# *Contours in triangle/tetrahedral cells*



Linear interpolation of cells implies piece-wise linear contours.

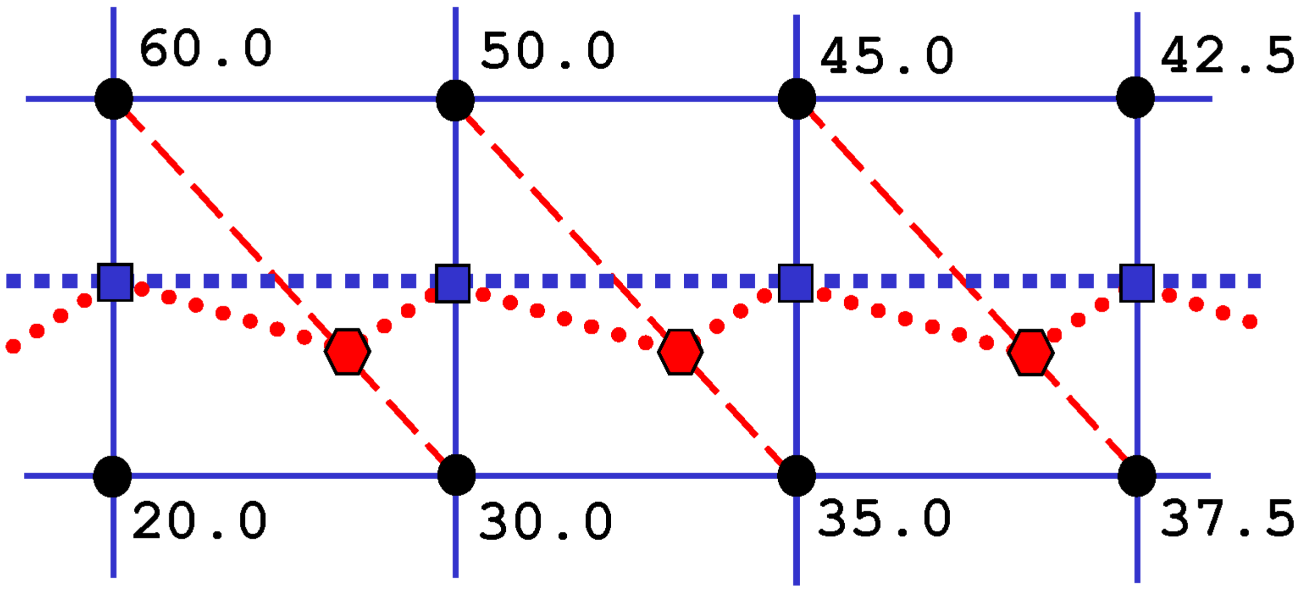
Contours are unambiguous, making "marching triangles" even simpler than "marching squares".

Question: Why not split quadrangles into two triangles (and hexahedra into five or six tetrahedra) and use marching triangles (tetrahedra)?

Answer: This can introduce periodic artifacts!

# *Contours in triangle/tetrahedral cells*

Illustrative example: Find contour at level *c*=40.0 !

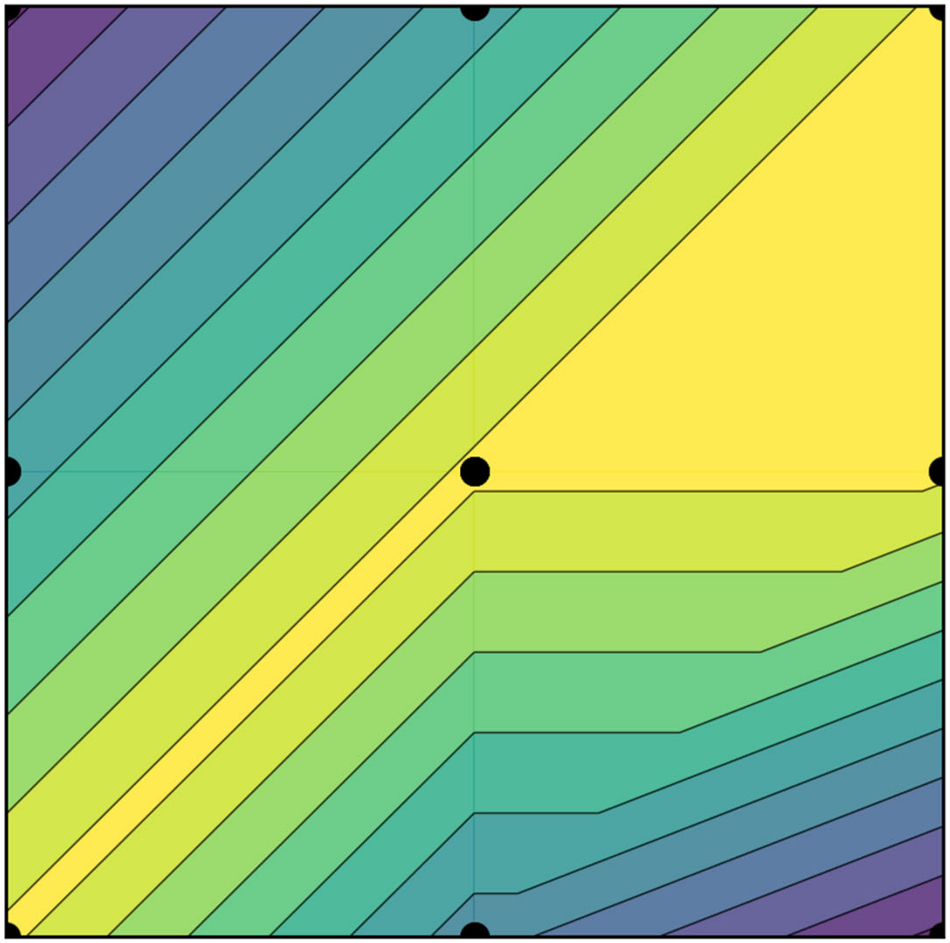



original quad grid, yielding vertices ■ and contour ■■■■■■
triangulated grid, yielding vertices ⬢ and contour ••••••

# Bi-Linear Interpolation: Comparisons

linear

(2 triangles per quad;
diagonal:
bottom-left,
top-right)



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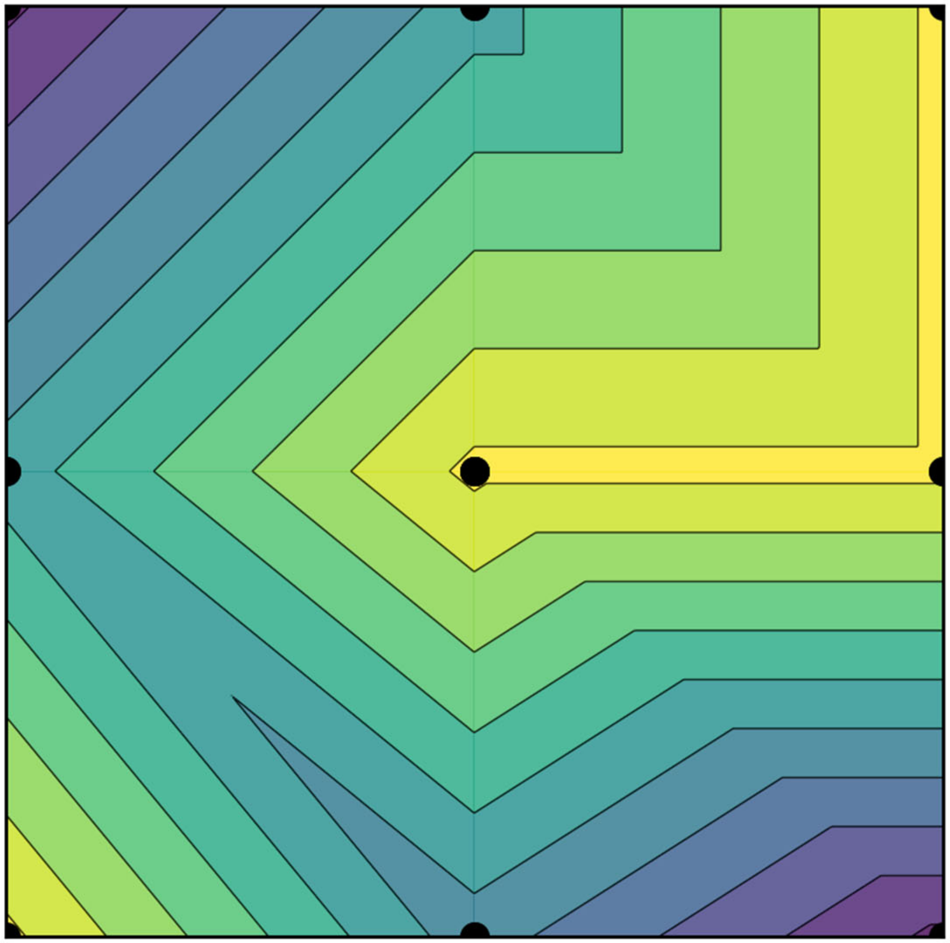
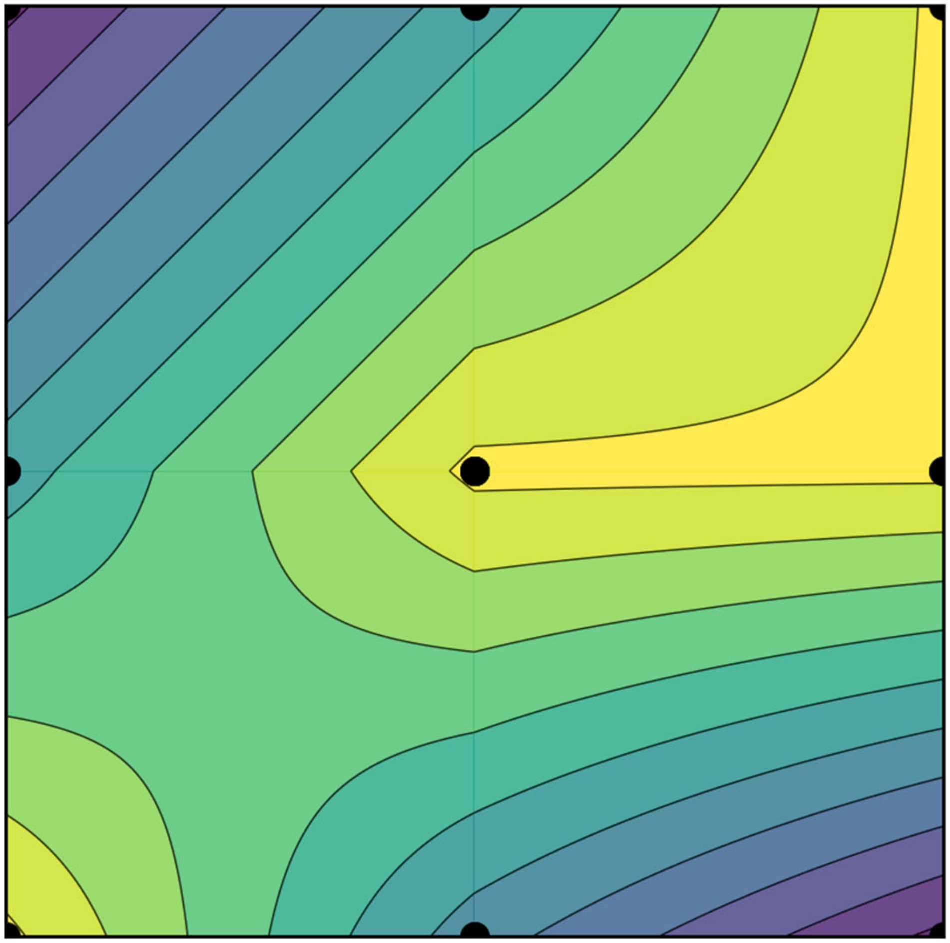


# Bi-Linear Interpolation: Comparisons

bi-linear



# Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
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- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama