CS 247 – Scientific Visualization
Lecture 27: Vector / Flow Visualization, Pt. 9

Markus Hadwiger, KAUST

Reading Assignment #15++

Read (optional):

- Tobias Günther, Irene Baeza Rojo:

  Introduction to Vector Field Topology

  https://cgl.ethz.ch/Downloads/Publications/Papers/2020/Gun20b/Gun20b.pdf

- Roxana Bujack, Lin Yan, Ingrid Hotz, Christoph Garth, Bei Wang:

  State of the Art in Time-Dependent Flow Topology: Interpreting Physical Meaningfulness Through Mathematical Properties

  https://onlinelibrary.wiley.com/doi/epdf/10.1111/cgf.14037

- B. Jobard, G. Erlebacher, M. Y. Hussaini:

  Lagrangian-Eulerian Advection of Noise and Dye Textures for Unsteady Flow Visualization

  http://dx.doi.org/10.1109/TVCG.2002.1021575

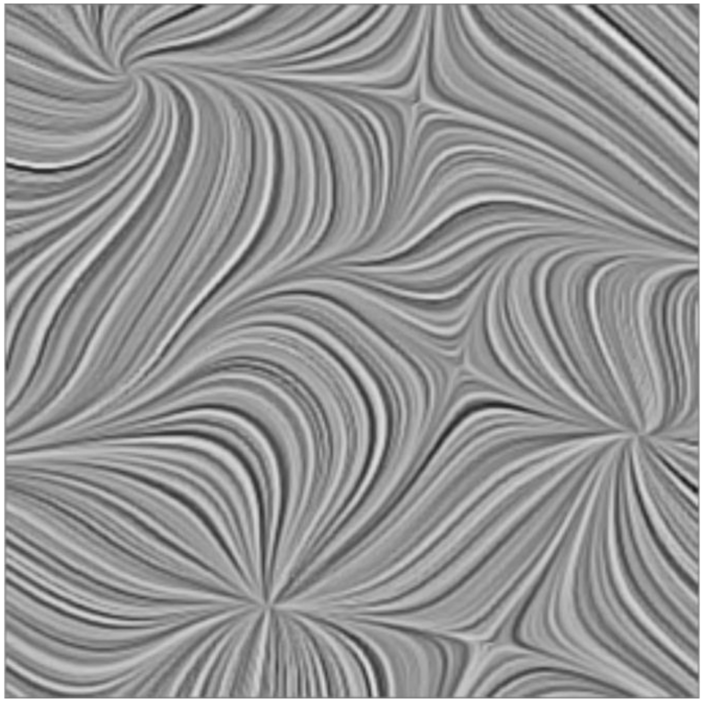
- Anna Vilanova, S. Zhang, Gordon Kindlmann, David Laidlaw:

  An Introduction to Visualization of Diffusion Tensor Imaging and Its Applications

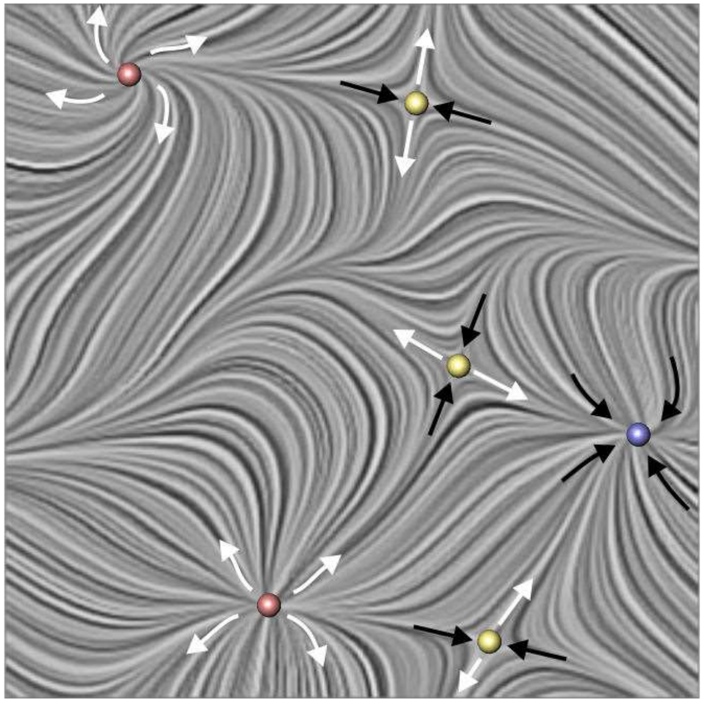
  http://vis.cs.brown.edu/docs/pdf/Vilanova-2005-IVD.pdf

Critical Points (Steady Flow!)

Classify critical points according to the *eigenvalues* of the velocity gradient tensor at the critical point



stream lines (LIC)



critical points (**v** = 0)

(Non-Linear) Dynamical Systems

Start with system of linear ODEs (with constant coefficients)

- Non-linear systems can be linearized around critical points
- Use linearization for characterization

$\dot{\mathbf{x}} = A\mathbf{x}$ $\quad$ A is an $n \times n$ matrix $\quad \longrightarrow$

$$\mathbf{v} = A\mathbf{x}, \quad \nabla\mathbf{v} = A.$$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

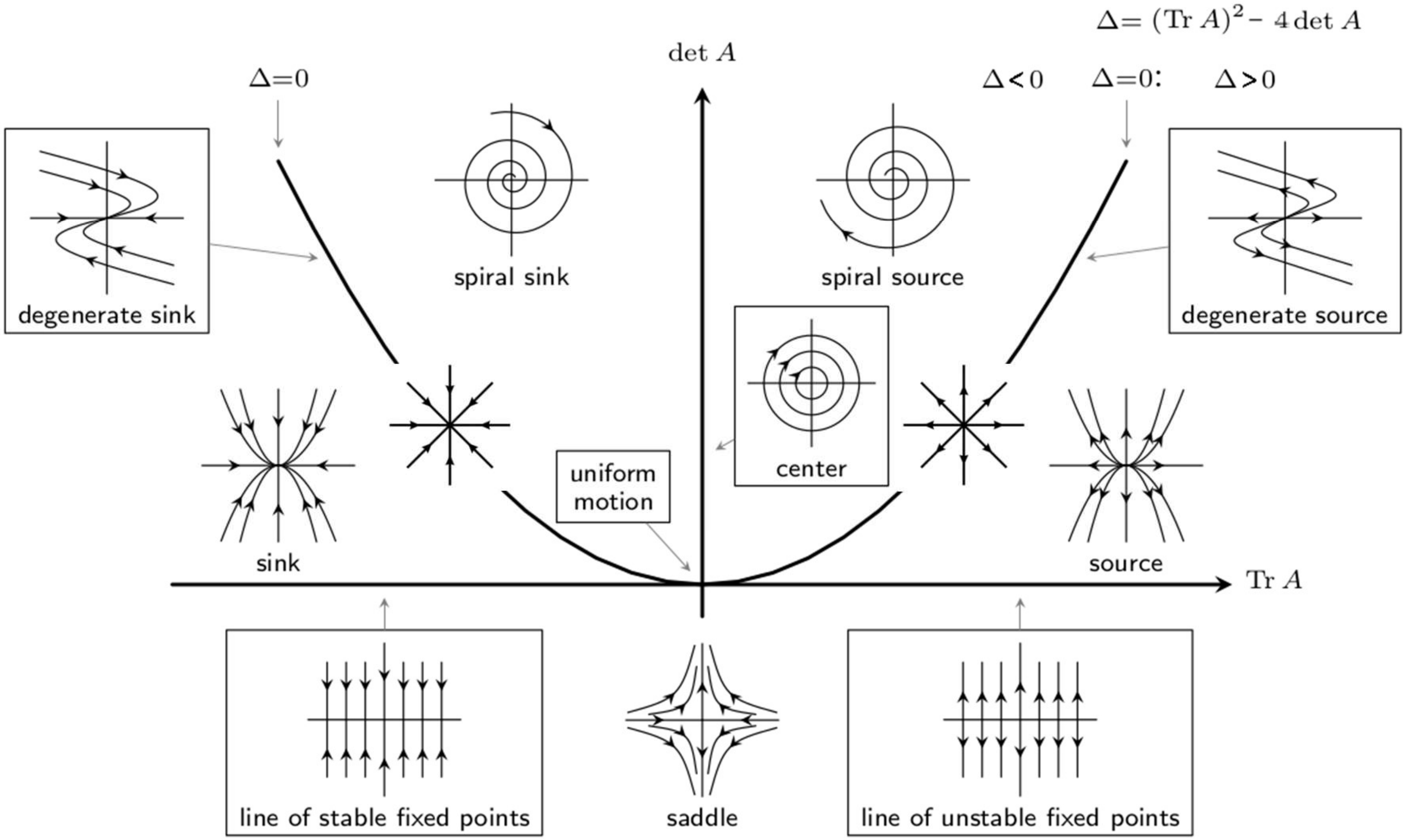
$$\mathbf{x}(0) = \mathbf{x}_0$$

solution: $\mathbf{x}(t) = e^{At}\mathbf{x}_0$

characterize behavior
through eigenvalues of A

Critical Points (Steady Flow!)

Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \mathrm{Tr}\, A)$-plane



Matrix Exponentials

Defined via same power series as usual exponential

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots$$

Easy if *X* is diagonalizable

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Exponentials of anti-symmetric matrices are rotation matrices

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Matrix Exponentials

Defined via same power series as usual exponential

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

Easy if *X* is diagonalizable

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Complex eigenvalues lead to rotation

$$X = \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix} \qquad \exp(tX) = e^{at} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}$$

$$\lambda_{1,2} = a \pm \mathbf{i}\omega$$

Classification of Critical Points

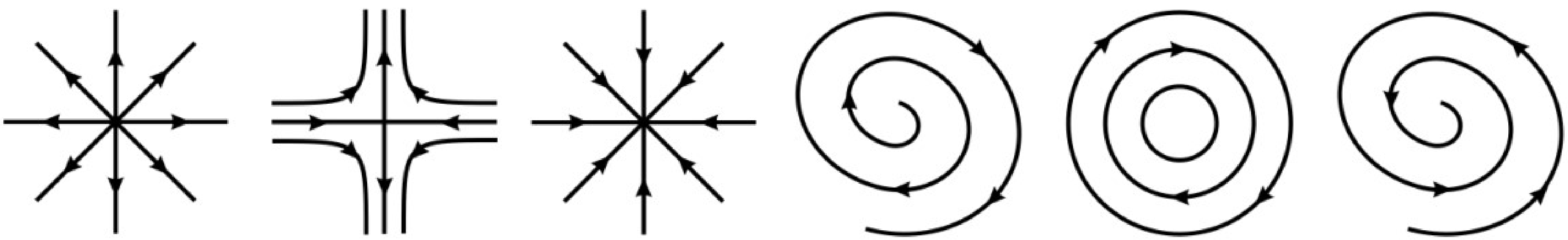
(Isolated) critical point (equilibrium point)

- Velocity vanishes (all components zero)

$$\mathbf{v}(\mathbf{x}_c) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_c \pm \epsilon) \neq \mathbf{0} \qquad\qquad \det(\, \nabla \mathbf{v}(\mathbf{x}_c)\,) \neq 0$$

Characterize using velocity gradient $\nabla \mathbf{v}$ at critical point $\mathbf{x}_c$

- Look at eigenvalues (and eigenvectors) of $\nabla \mathbf{v}$



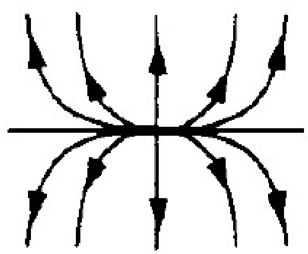
Repelling node	Saddle point	Attracting node	Repelling focus	Center	Attracting focus
$R_1, R_2 > 0$	$R_1 < 0, R_2 > 0$	$R_1, R_2 < 0$	$R_1 = R_2 > 0$	$R_1 = R_2 = 0$	$R_1 = R_2 < 0$
$I_1 = I_2 = 0$	$I_1 = I_2 = 0$	$I_1 = I_2 = 0$	$I_1 = -I_2 \neq 0$	$I_1 = -I_2 \neq 0$	$I_1 = -I_2 \neq 0$

the first three phase portraits are special cases, see later slides!

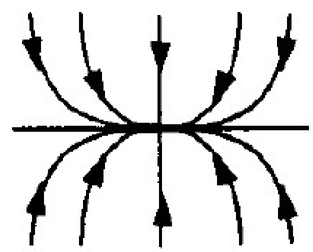
A Few Details (1)

Repelling/attracting nodes

- Do not necessarily imply that streamlines are straight lines (do not confuse with the linear system of ODEs!)
- They are only straight lines when both eigenvalues are real and have the same sign, *and are also equal* (as in the phase portraits before)
- If they are not equal:



Repelling Node
R1, R2 > 0
I1, I2 = 0

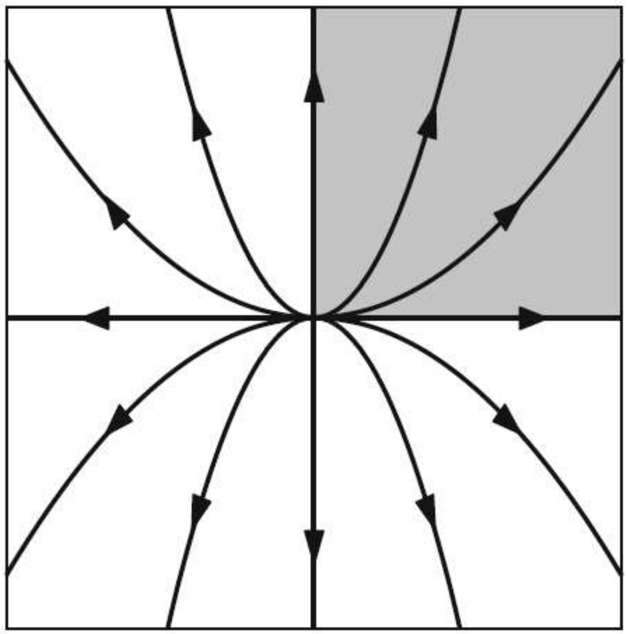


Attracting Node
R1, R2 < 0
I1, I2 = 0

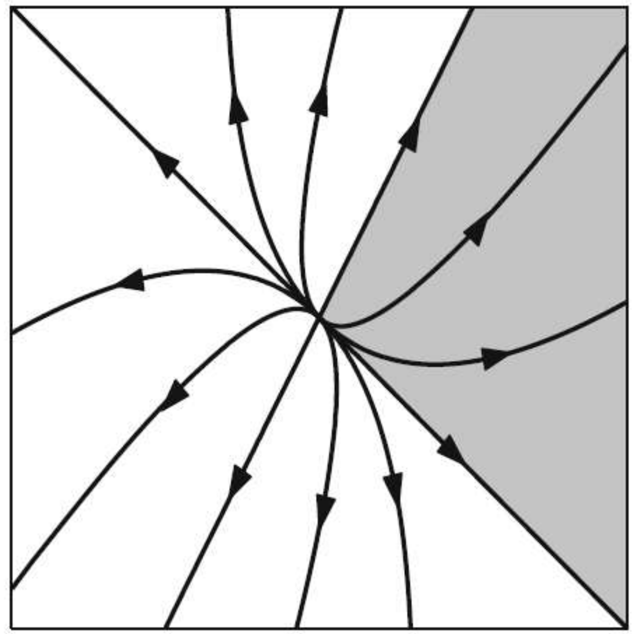
A Few Details (2)

What about skew axes?

- Both of the systems below have eigenvalues 3 and 6
- Jordan normal form (Jordan canonical form) gives details



$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$



$$\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}$$

Jordan Normal Form (2x2 Matrix)

For every real 2x2 matrix A there is an invertible P such that

$P^{-1}AP$ is one of the following Jordan matrices (all entries are real):

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$ (defective matrix)

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Each of these has its corresponding rule for constructing P

- Example on prev. slide (the two eigenvectors are not orthogonal):

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

See also *algebraic* and *geometric multiplicity* of eigenvalues

Jordan Normal Form (2x2 Matrix)

For every real 2x2 matrix A there is an invertible P such that

$P^{-1}AP$ is one of the following Jordan matrices (all entries are real):

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$ (defective matrix)

same eigenvalues, trace, determinant!

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Each of these has its corresponding rule for constructing P

- Example on prev. slide (the two eigenvectors are not orthogonal):

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

See also *algebraic* and *geometric multiplicity* of eigenvalues

Another Example

$P^{-1}AP$ has form J_1

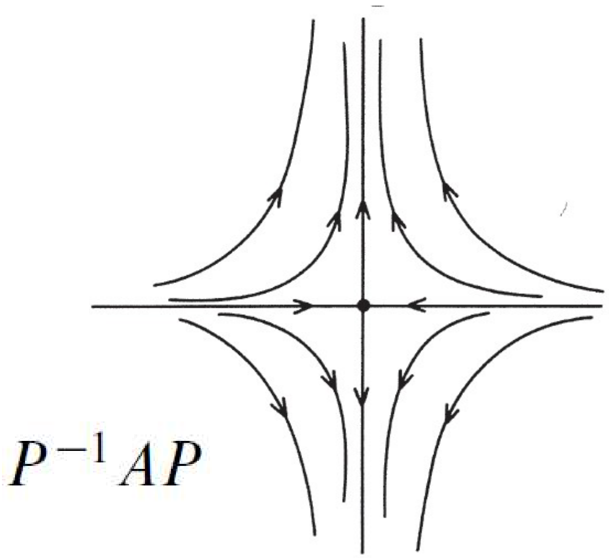
$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

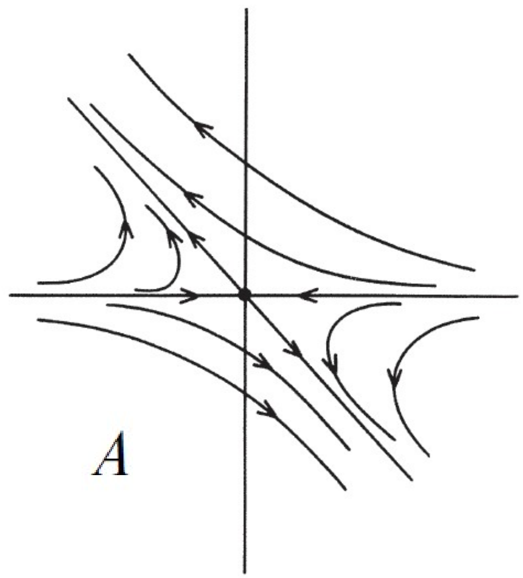
Eigenvalues:

$\lambda_1 = -1$

$\lambda_2 = 2$

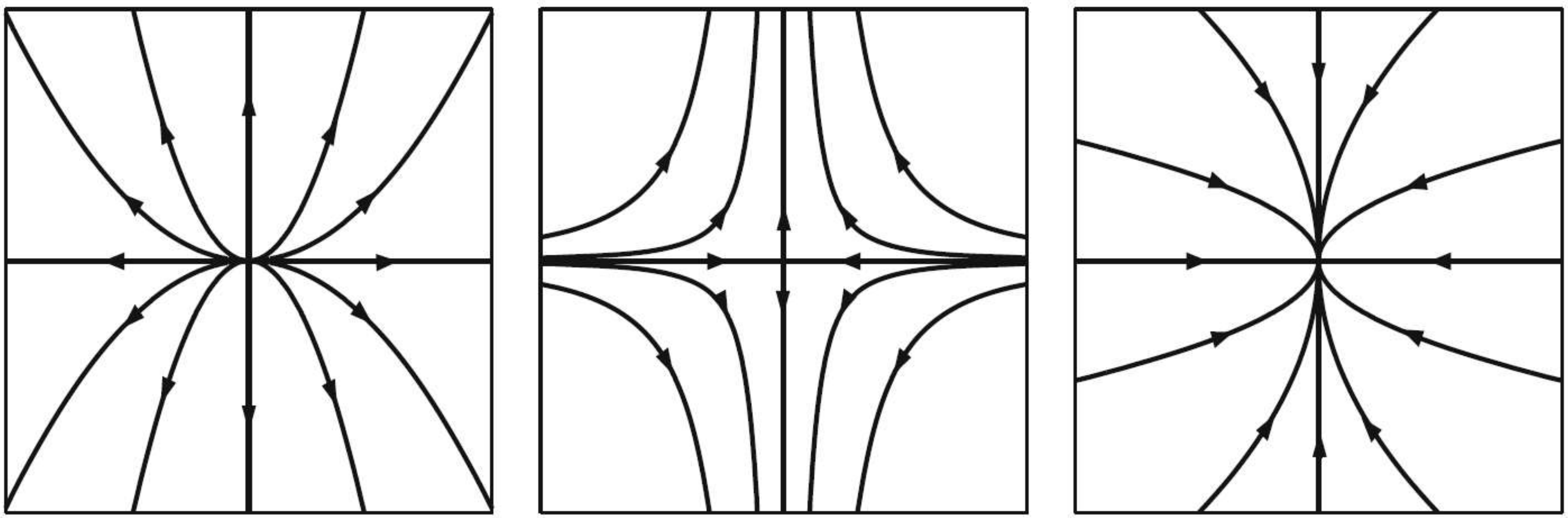
$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$





Jordan Form Characterization (1)

Phase portraits corresponding to Jordan matrix $J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$



$0 < \lambda_1 < \lambda_2$
unstable node

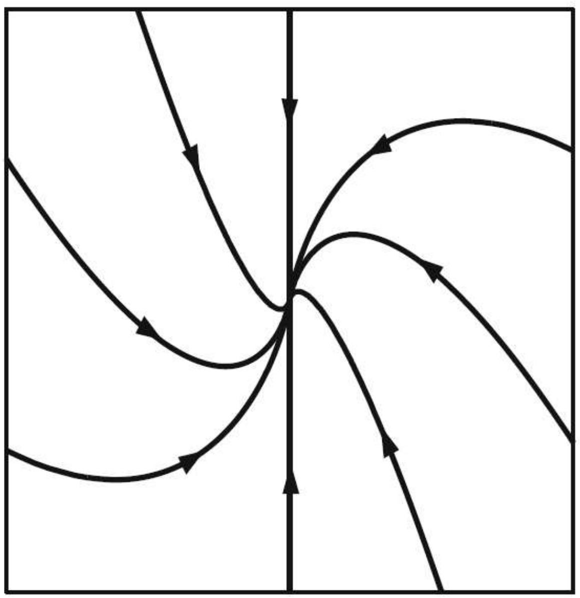
$\lambda_1 < 0 < \lambda_2$
saddle

$\lambda_1 < \lambda_2 < 0$
stable node

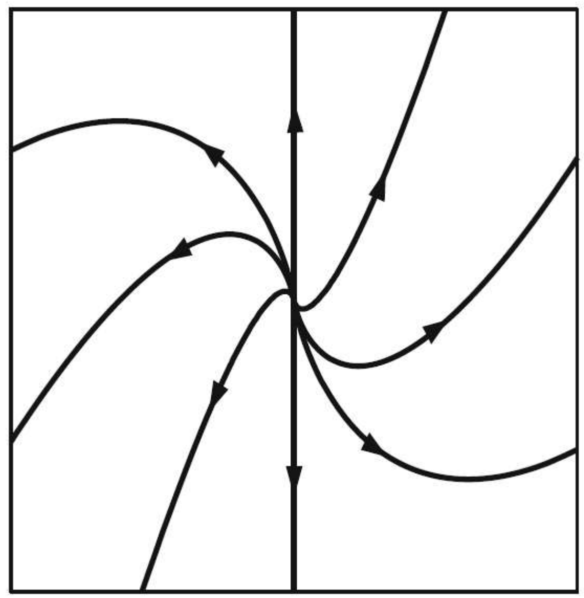
Jordan Form Characterization (2)

Phase portraits corresponding to Jordan matrix
(matrix is defective: eigenspaces collapse,
geometric multiplicity less than algebraic multiplicity)

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$



$\lambda < 0$
stable improper node

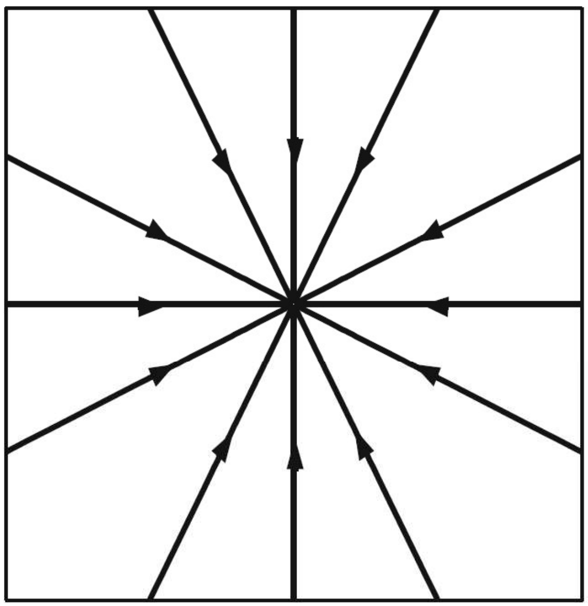


$\lambda > 0$
unstable improper node

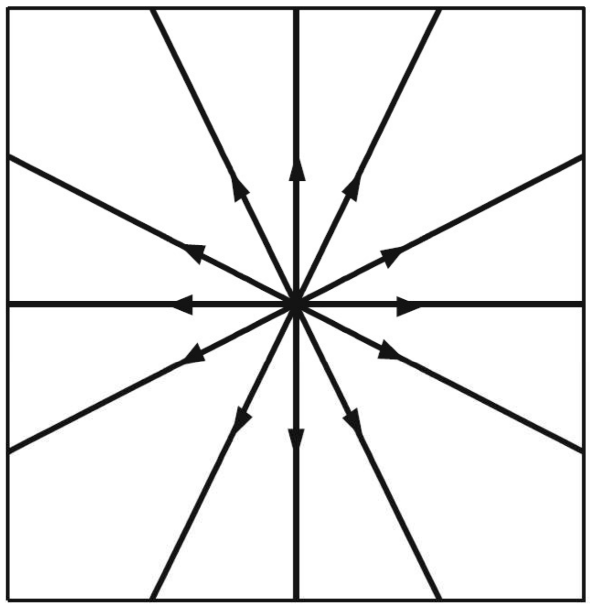
Jordan Form Characterization (3)

Phase portraits corresponding to Jordan matrix

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$



$\lambda < 0$
stable star node

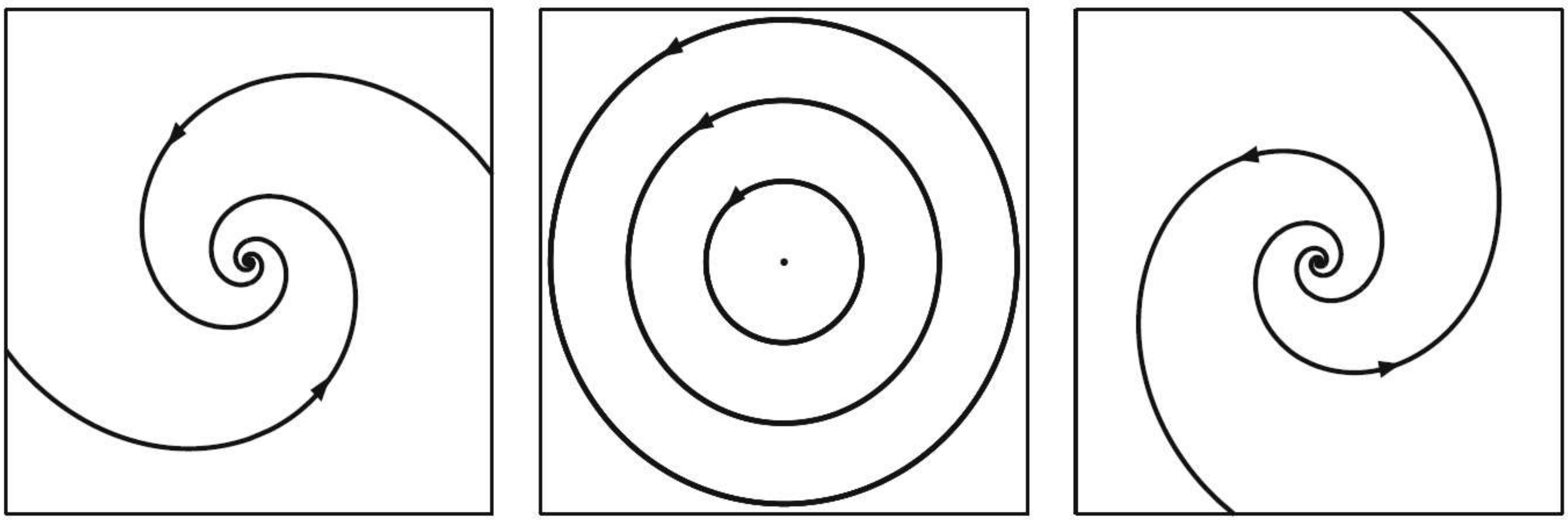


$\lambda > 0$
unstable star node

Jordan Form Characterization (4)

Phase portraits corresponding to Jordan matrix

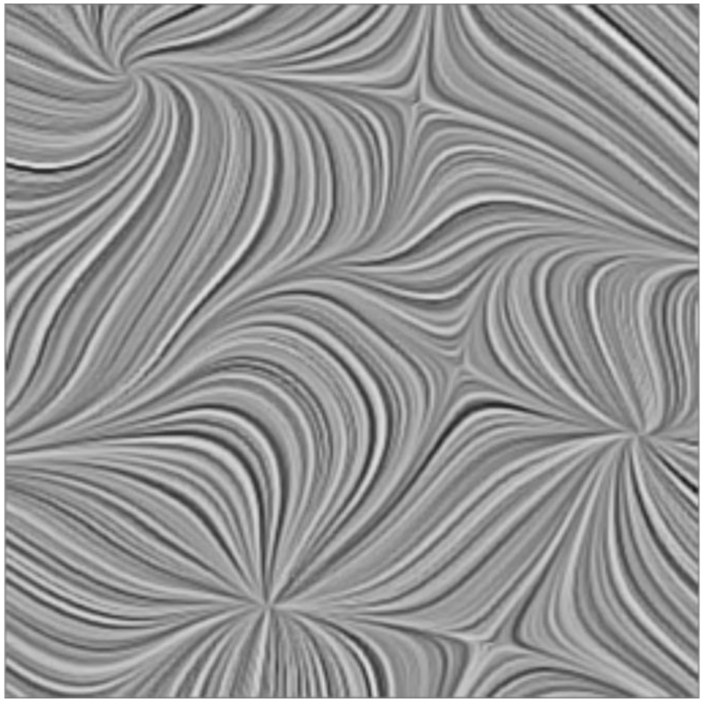
$$J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$



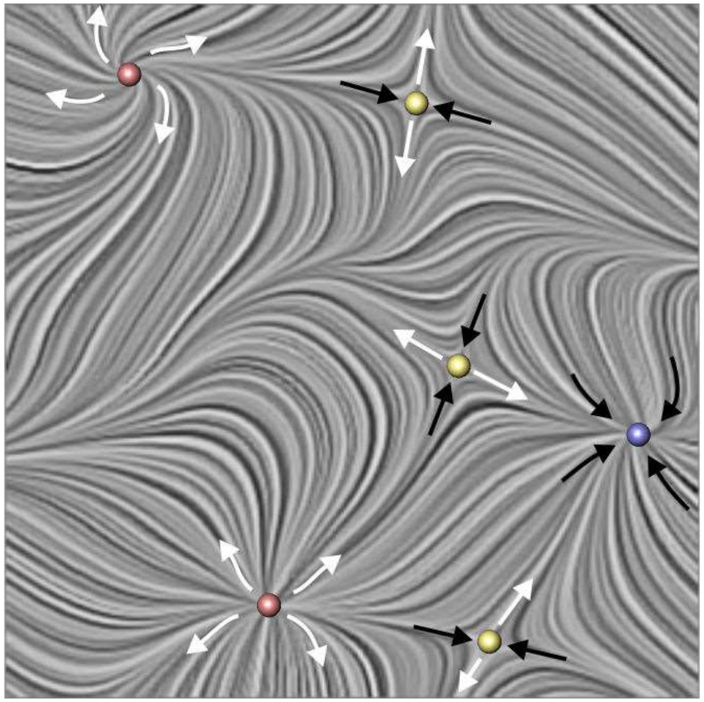
$a < 0$ stable spiral node | $a = 0$ center | $a > 0$ unstable spiral node

Critical Points (Steady Flow!)

Classify critical points according to the *eigenvalues* of the velocity gradient tensor at the critical point



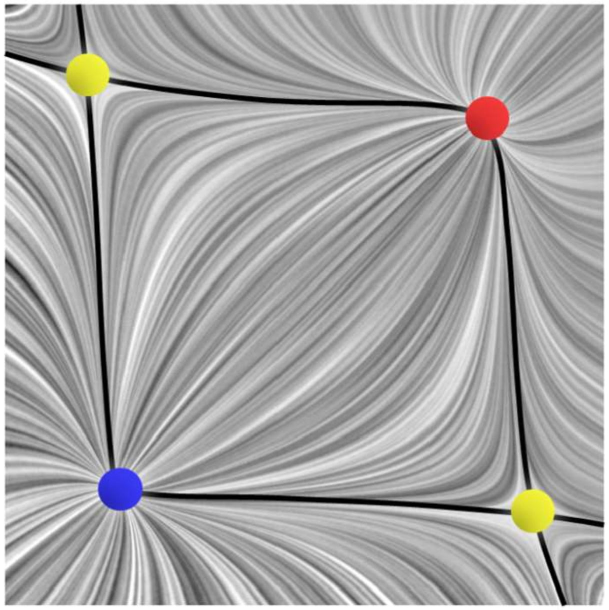
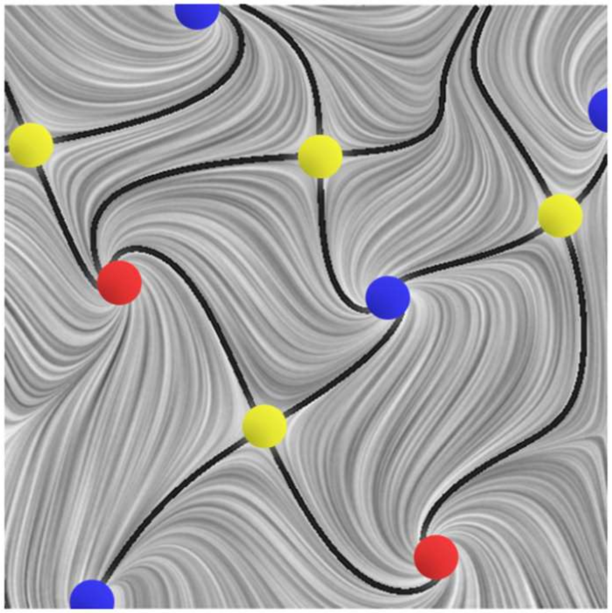
stream lines (LIC)



critical points (**v** = 0)

Vector Field Topology: Topological Skeleton

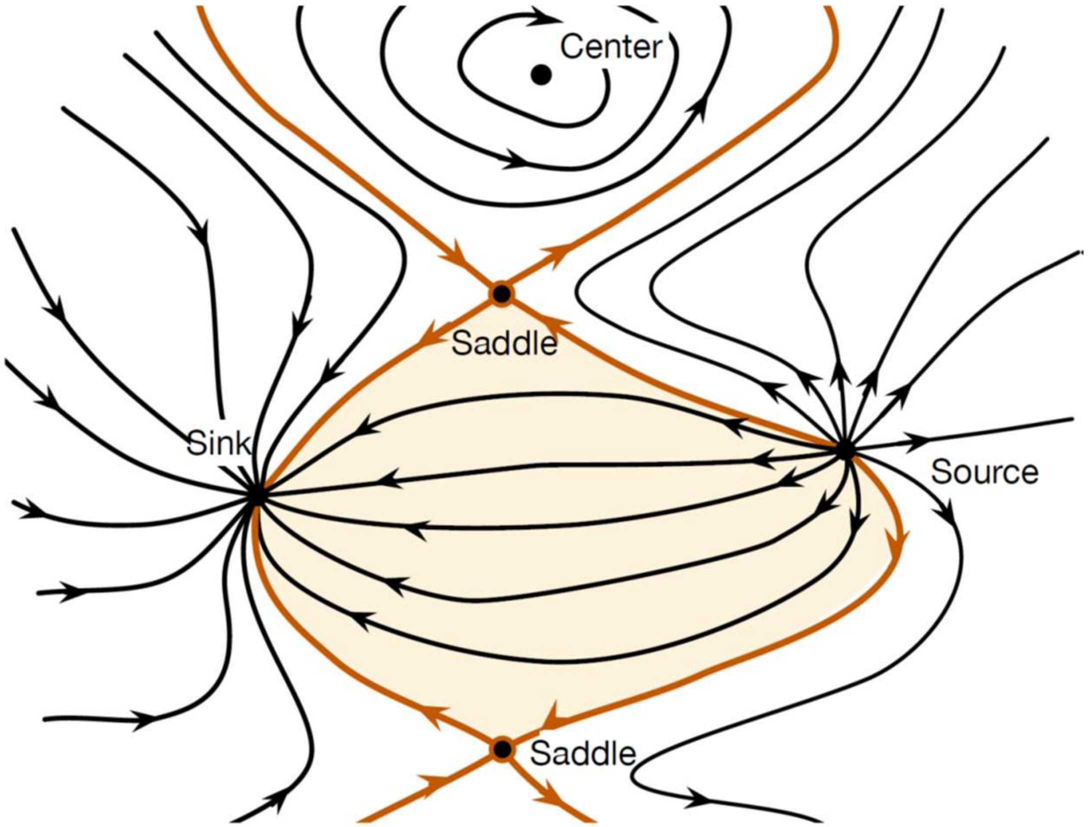
Connect critical points by *separatrices*

Sources (red), sinks (blue), saddles (yellow)

Vector Field Topology: Topological Skeleton

Connect critical points by *separatrices*



Index of Critical Points / Vector Fields

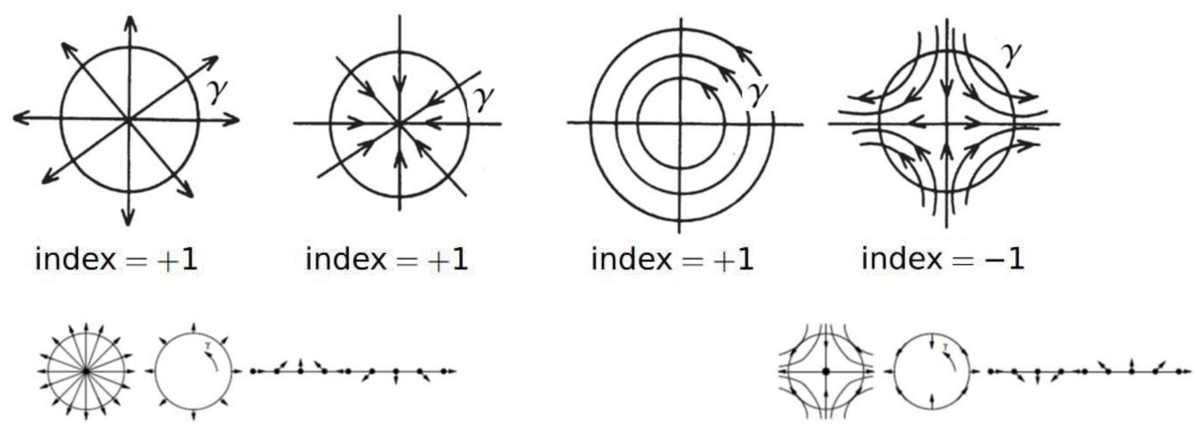
Poincaré index (in scalar field topology we had the *Morse index*)

- Can compute index (winding number) for each critical point
- Index of a region is the sum of the critical point indexes inside
- Sum of all indexes over a manifold is its Euler characteristic

Do a loop (Jordan curve) around each critical point: the index is its (Brouwer) degree: integer how often the vector field along the loop turns around (determined by angle 1-form integrated over oriented 1-manifold)

$$\text{index}_\gamma = \frac{1}{2\pi} \oint_\gamma d\alpha$$

$$\alpha = \arctan \frac{v}{u}$$

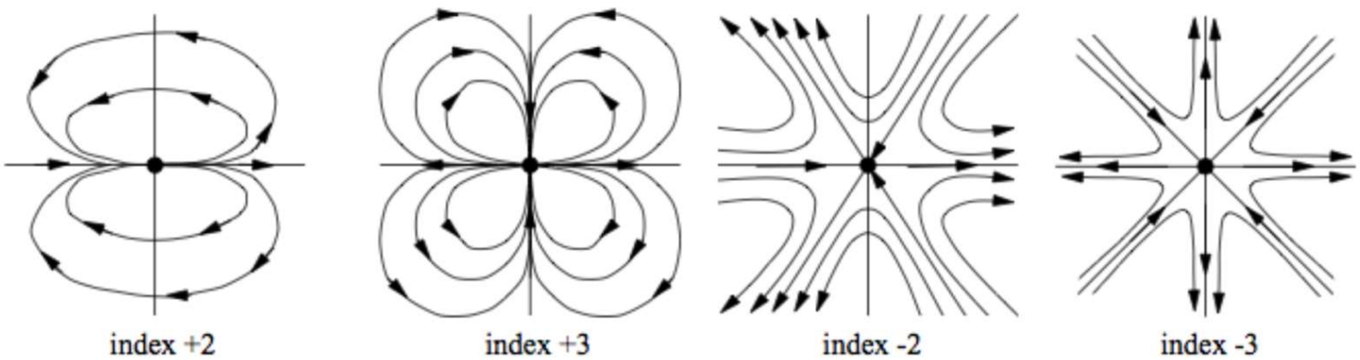


Higher-Order Critical Points

Higher than first-order

- Sectors can by elliptic, parabolic, hyperbolic
- For index sum over number of elliptic and hyperbolic sectors

$$\text{index}_{cp} = 1 + \frac{n_e - n_h}{2}$$



(dipole)

(see monkey saddle)

Example: Differential Topology

Topological information from vector fields on manifold

- Independent of actual vector field! Poincaré-Hopf theorem
- Useful constraints: vector field editing, simplification, sphere always has critical point, …

Topological invariant: Euler characteristic $\chi(M)$ of manifold M

(for 2-manifold mesh: $\chi(M) = V - E + F$)

$\chi = 2 - 2g$ (orientable)

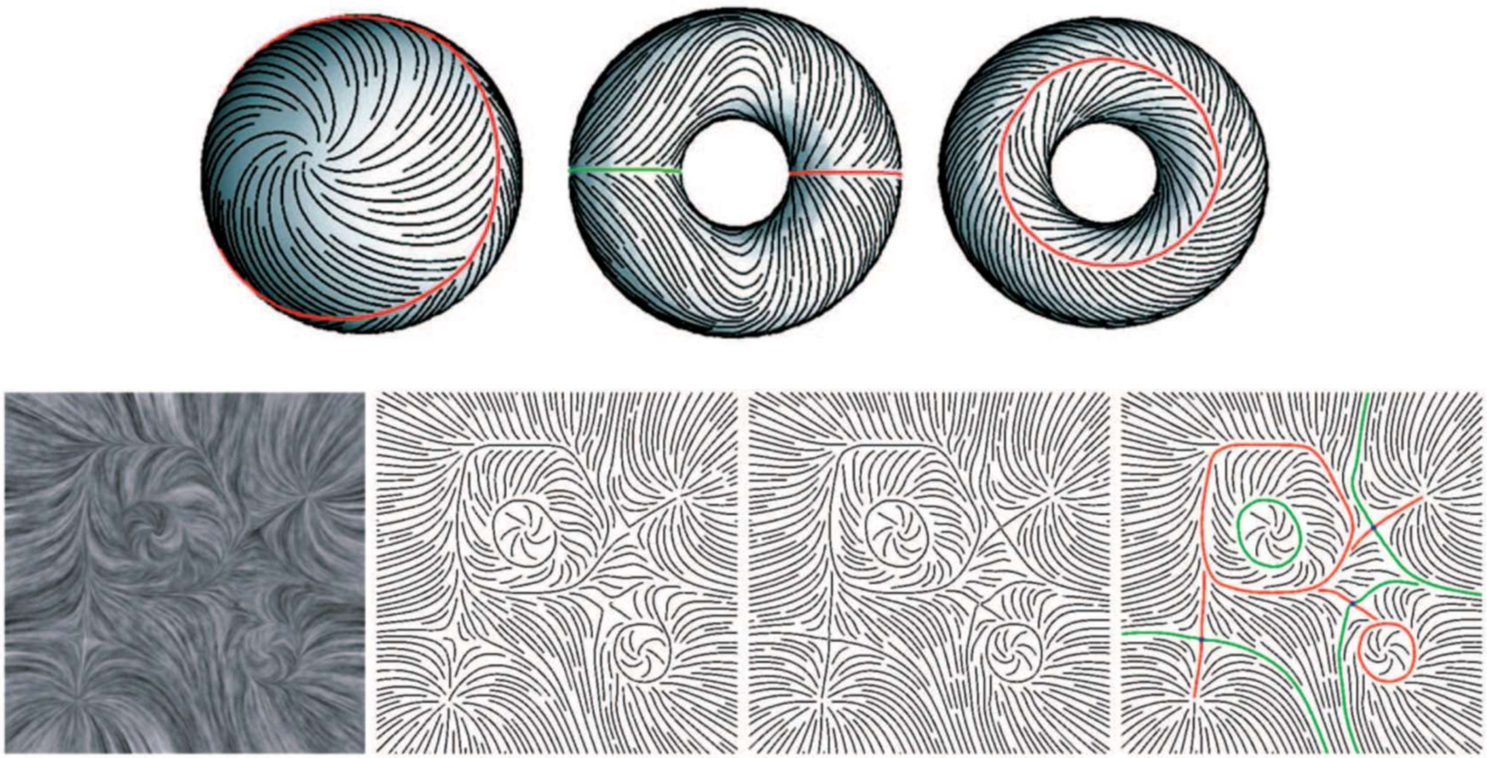
genus $g = 0$
Euler characteristic $\chi = 2$

genus $g = 1$
Euler characteristic $\chi = 0$

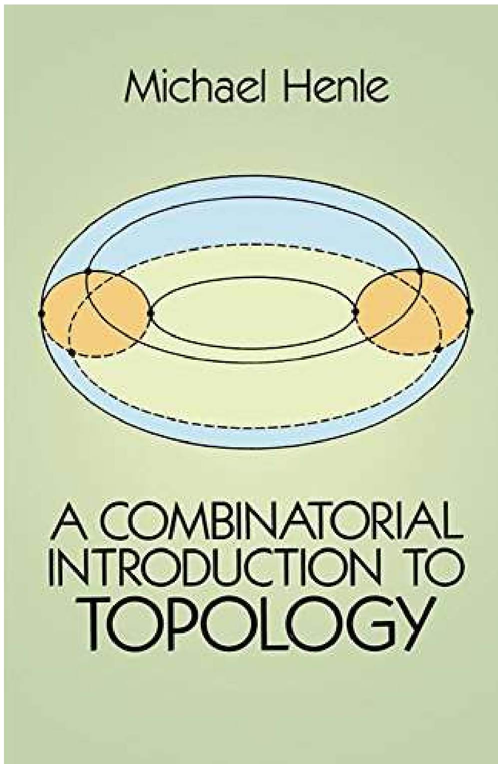
genus $g = 2$
Euler characteristic $\chi = -2$

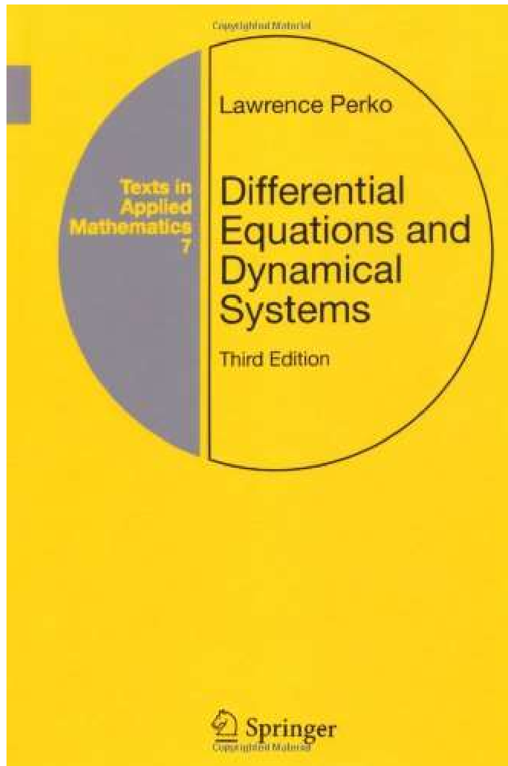
Example: Vector Field Editing

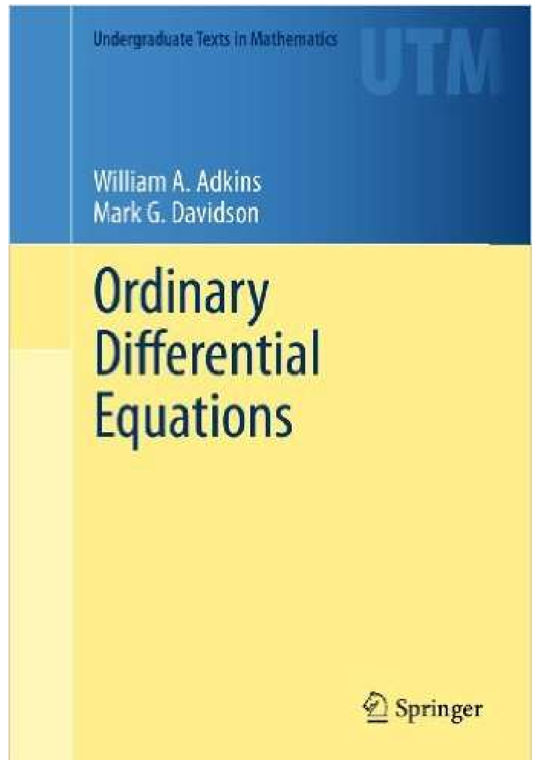
Guoning Chen et al., Vector Field Editing and Periodic Orbit Extraction Using Morse Decomposition, IEEE TVCG, 2007



Recommended Books







Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
- Daniel Weiskopf
- Torsten Möller
- Ronny Peikert
- Philipp Muigg
- Christof Rezk-Salama