CS 247 – Scientific Visualization
Lecture 26: Vector / Flow Visualization, Pt. 8

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Reading Assignment #15++

Read (optional):

- Tobias Günther, Irene Baeza Rojo:

  Introduction to Vector Field Topology

  https://cgl.ethz.ch/Downloads/Publications/Papers/2020/Gun20b/Gun20b.pdf

- Roxana Bujack, Lin Yan, Ingrid Hotz, Christoph Garth, Bei Wang:

  State of the Art in Time-Dependent Flow Topology: Interpreting Physical Meaningfulness Through Mathematical Properties

  https://onlinelibrary.wiley.com/doi/epdf/10.1111/cgf.14037

- B. Jobard, G. Erlebacher, M. Y. Hussaini:

  Lagrangian-Eulerian Advection of Noise and Dye Textures for Unsteady Flow Visualization

  http://dx.doi.org/10.1109/TVCG.2002.1021575

- Anna Vilanova, S. Zhang, Gordon Kindlmann, David Laidlaw:

  An Introduction to Visualization of Diffusion Tensor Imaging and Its Applications

  http://vis.cs.brown.edu/docs/pdf/Vilanova-2005-IVD.pdf

Quiz #4: May 5

Organization

- First 30 min of lecture
- No material (book, notes, ...) allowed

Content of questions

- Lectures (both actual lectures and slides)
- Reading assignments (except optional ones)
- Programming assignments (algorithms, methods)
- Solve short practical examples

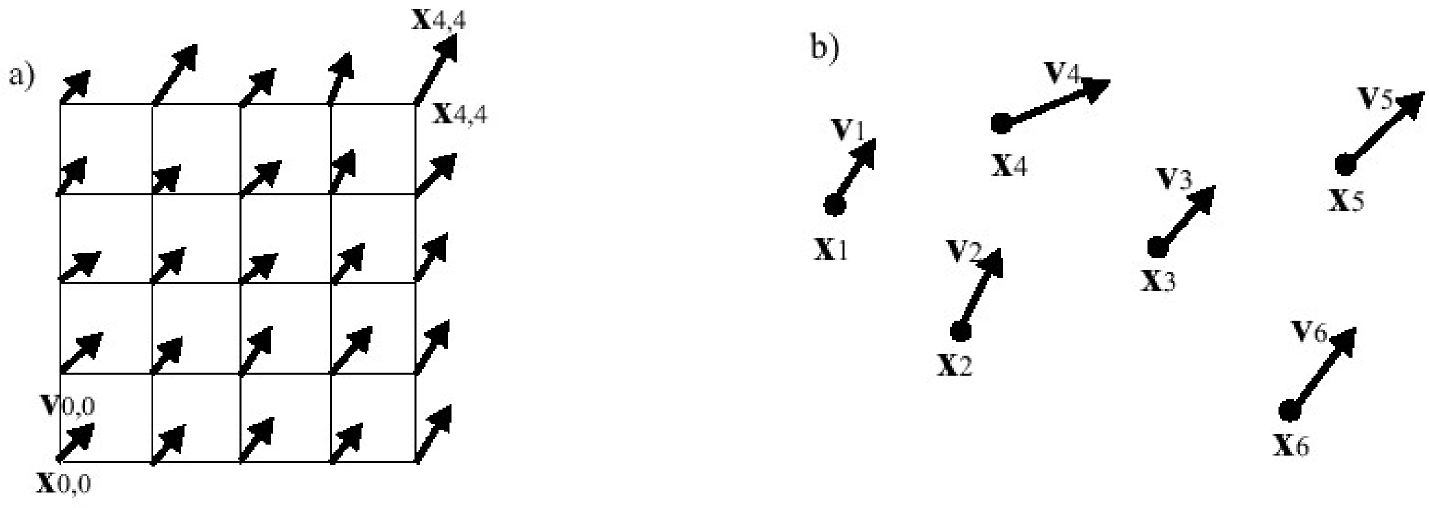
Lagrangian vs. Eulerian

Eulerian

- Flow properties given at fixed spatial positions (grid points)
- Partial time derivative

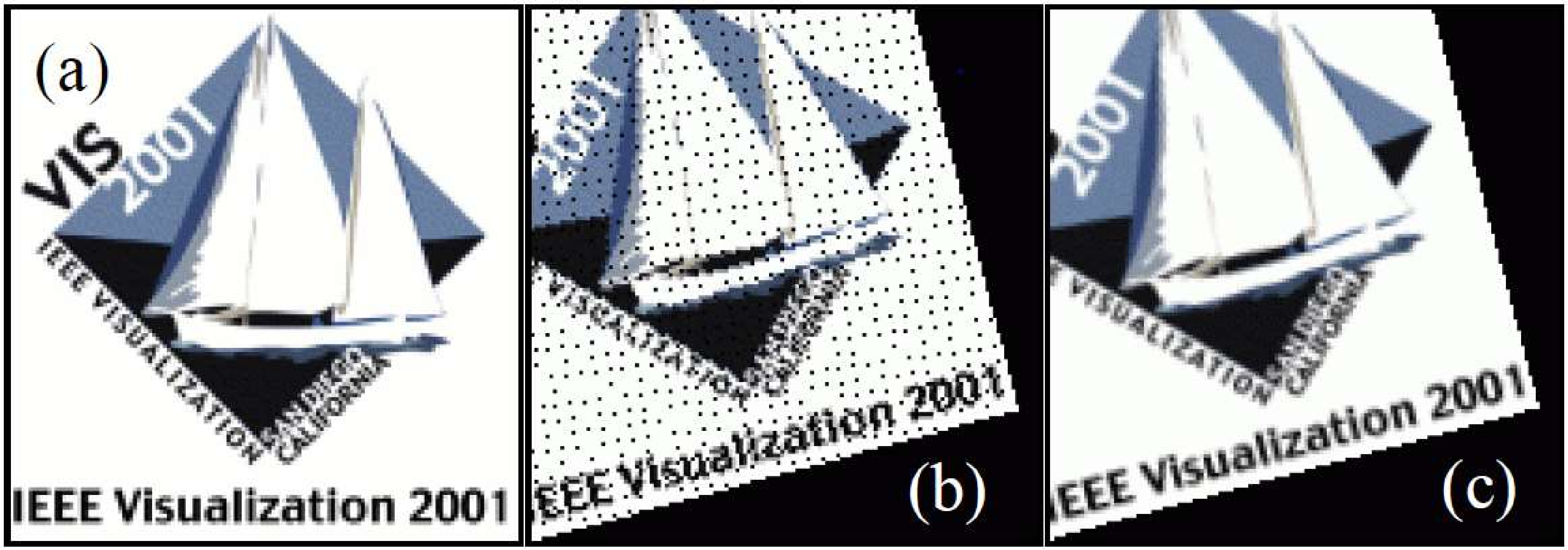
Lagrangian

- Flow properties given for each particle (particles are moving)
- Material time derivative



Lagrangian vs. Eulerian

- Lagrangian: move along with the particle
- Eulerian: consider fixed point in space, look at particles moving through



- Example for pixels: rotate image (a),
  Lagrangian: move pixels forward (b),
  Eulerian: fetch pixels from backward direction (c)

Material Derivative (1)

The material time derivative (convective derivative) gives the rate of change when following a particle in the flow

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

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$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)\, T$$

Material Derivative (1)

The material time derivative (convective derivative) gives the rate of change when following a particle in the flow

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}$$

Material Derivative (2)

Actually, nothing else than application of the multi-variable chain rule:

$$dT = \frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

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$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

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$$u := \frac{dx}{dt}, \qquad v := \frac{dy}{dt}, \qquad w := \frac{dz}{dt}$$

Material Derivative (2)

Actually, nothing else than application of the multi-variable chain rule:

We are given $T(x, y, z, t)$ with four independent variables;

But now we want to go along a parameterized path with parameter t, so x, y, z become dependent variables: $x(t),\ \ y(t),\ \ z(t)$

Along this path, our goal is now to compute the derivative of the function

$T\left(x(t), y(t), z(t), t\right)$ with t as only independent variable:

$$\frac{d}{dt} T\left(x(t), y(t), z(t), t\right) =$$

$$\frac{\partial}{\partial t} T(x, y, z, t) + \frac{\partial}{\partial x} T(x, y, z, t) \frac{d}{dt} x(t) + \frac{\partial}{\partial y} T(x, y, z, t) \frac{d}{dt} y(t) + \frac{\partial}{\partial z} T(x, y, z, t) \frac{d}{dt} z(t)$$

$$u(t) := \frac{dx(t)}{dt}, \qquad v(t) := \frac{dy(t)}{dt}, \qquad w(t) := \frac{dz(t)}{dt}$$

Advection

Advection equation; velocity field **u(** *x, y, z, t* **)**,
no change following particle, just advection:
set material derivative = 0:

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = 0$$

In the Navier-Stokes equations: “self-advection” of velocity

- Advect scalar components of velocity field individually (actually two equations in 2D, three equations in 3D)

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u}$$

this is equivalent to saying that the acceleration is zero!

Vector Fields and Dynamical Systems (1)

Velocity gradient tensor, (vector field → tensor field)

- Gradient of vector field: how does the vector field change?
- In Cartesian coordinates: *spatial partial derivatives (Jacobian matrix)*

$$\nabla \mathbf{v}\,(x,y,z) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

these are partial derivatives!

- Can be decomposed into *symmetric* part + *anti-symmetric* part

$$\nabla \mathbf{v} = \mathbf{D} + \mathbf{S}$$ *velocity gradient tensor*

sym.: $\mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^{T} \right)$ deform.: *rate-of-strain tensor*

skew-sym.: $\mathbf{S} = \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^{T} \right)$ rotation: *vorticity/spin tensor*

Vector Fields and Dynamical Systems (2)

Vorticity/spin/angular velocity tensor

- Antisymmetric part of velocity gradient tensor
- Corresponds to vorticity/curl/angular velocity (beware of factor ½)

these are partial derivatives!

$$\mathbf{S} = ½ \left(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathrm{T}} \right)$$

$$\mathbf{S} = ½ \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{v} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

$\mathbf{S}$ acts on vector like cross product with ω : $\mathbf{S}\,\bullet\; = ½\,\omega \times$

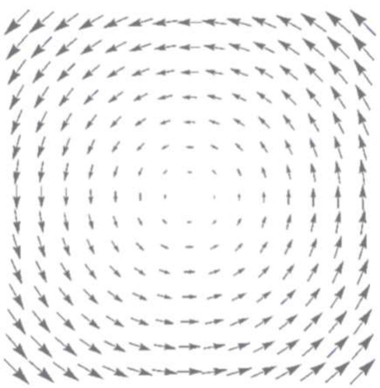
$$\mathbf{v}^{(r)} = \mathbf{S} \cdot d\mathbf{r} = \frac{1}{2}\left[\nabla\mathbf{v} - (\nabla\mathbf{v})^T\right] \cdot d\mathbf{r} = \frac{1}{2}\boldsymbol{\omega} \times d\mathbf{r}$$

Angular Velocity of Rigid Body Rotation

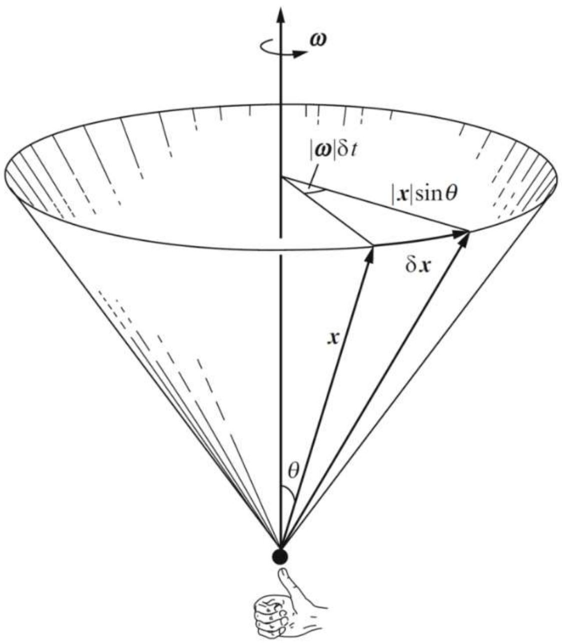
Rate of rotation

- Scalar ω: angular displacement per unit time (rad s^{-1})
  - Angle Θ at time t is $\Theta(t) = \omega t$; $\omega = 2\pi f$ where f is the frequency ($f = 1/T$; s^{-1})
- Vector $\boldsymbol{\omega}$: axis of rotation; magnitude is angular speed (if $\boldsymbol{\omega}$ is curl: speed x2)
  - Beware of different conventions that differ by a factor of ½ !

Cross product of ½$\boldsymbol{\omega}$ with vector to center of rotation ($\mathbf{r}$) gives linear velocity vector $\mathbf{v}$ (tangent)



$$\mathbf{v}^{(r)} = \frac{1}{2}\boldsymbol{\omega} \times d\mathbf{r}$$



Velocity Gradient Tensor and Components (1)

Velocity gradient tensor

(here: in Cartesian coordinates)

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial}{\partial x} v^x & \frac{\partial}{\partial y} v^x & \frac{\partial}{\partial z} v^x \\ \frac{\partial}{\partial x} v^y & \frac{\partial}{\partial y} v^y & \frac{\partial}{\partial z} v^y \\ \frac{\partial}{\partial x} v^z & \frac{\partial}{\partial y} v^z & \frac{\partial}{\partial z} v^z \end{bmatrix}$$

these are the same partial derivatives as before!

$$\nabla \mathbf{v} = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right) + \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^T\right)$$

Velocity Gradient Tensor and Components (2)

Rate-of-strain (rate-of-deformation) tensor

(symmetric part; here: in Cartesian coordinates)

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} 2\frac{\partial}{\partial x}v^x & \frac{\partial}{\partial y}v^x + \frac{\partial}{\partial x}v^y & \frac{\partial}{\partial z}v^x + \frac{\partial}{\partial x}v^z \\ \frac{\partial}{\partial x}v^y + \frac{\partial}{\partial y}v^x & 2\frac{\partial}{\partial y}v^y & \frac{\partial}{\partial z}v^y + \frac{\partial}{\partial y}v^z \\ \frac{\partial}{\partial x}v^z + \frac{\partial}{\partial z}v^x & \frac{\partial}{\partial y}v^z + \frac{\partial}{\partial z}v^y & 2\frac{\partial}{\partial z}v^z \end{bmatrix}$$

$$tr(\mathbf{D}) = \nabla \cdot \mathbf{v}$$

Velocity Gradient Tensor and Components (3)

Vorticity tensor (spin tensor)

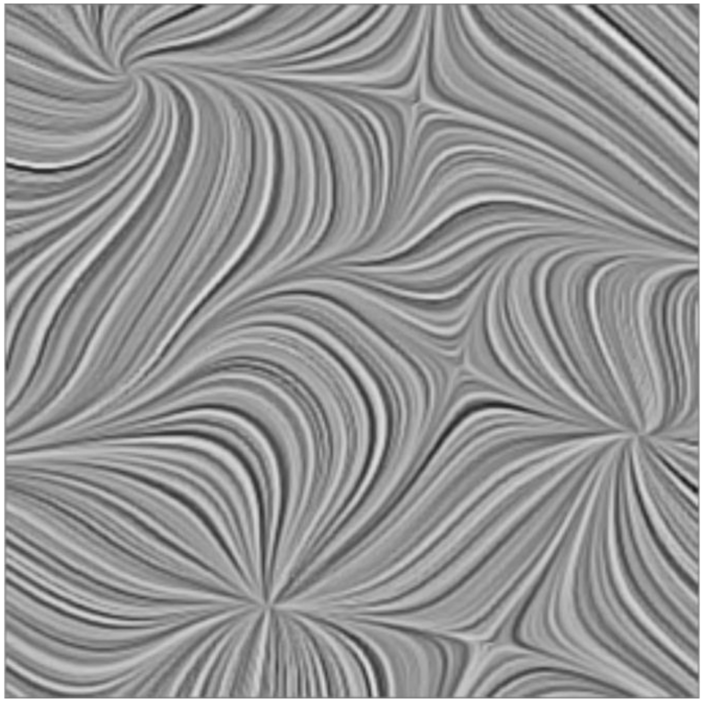
(skew-symmetric part; here: in Cartesian coordinates)

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial}{\partial y}v^x - \frac{\partial}{\partial x}v^y & \frac{\partial}{\partial z}v^x - \frac{\partial}{\partial x}v^z \\ \frac{\partial}{\partial x}v^y - \frac{\partial}{\partial y}v^x & 0 & \frac{\partial}{\partial z}v^y - \frac{\partial}{\partial y}v^z \\ \frac{\partial}{\partial x}v^z - \frac{\partial}{\partial z}v^x & \frac{\partial}{\partial y}v^z - \frac{\partial}{\partial z}v^y & 0 \end{bmatrix}$$

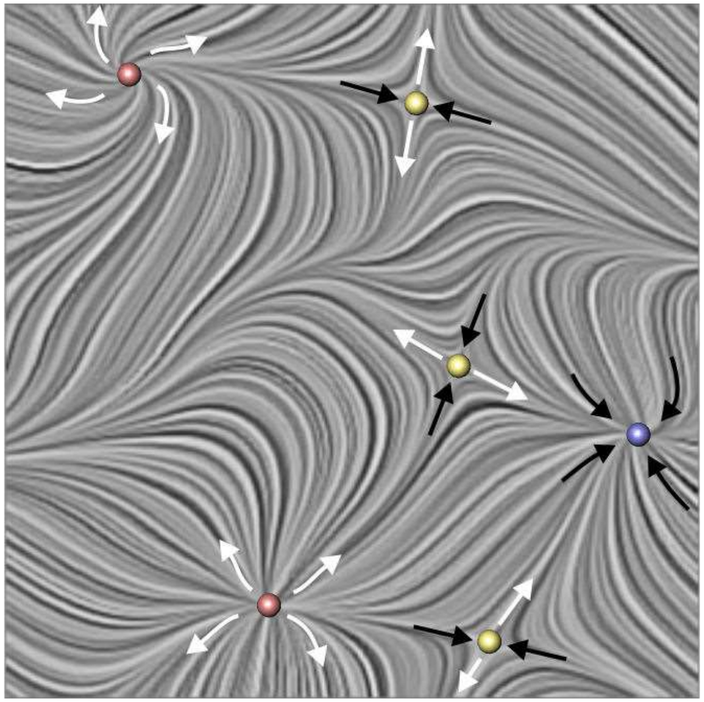
$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \qquad \boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$$

Critical Points (Steady Flow!)

Classify critical points according to the *eigenvalues* of the velocity gradient tensor at the critical point



stream lines (LIC)



critical points ($\mathbf{v} = 0$)

(Non-Linear) Dynamical Systems

Start with system of linear ODEs (with constant coefficients)

- Non-linear systems can be linearized around critical points
- Use linearization for characterization

$\dot{\mathbf{x}} = A\mathbf{x}$ A is an $n \times n$ matrix → $\mathbf{v} = A\mathbf{x},$ $\nabla \mathbf{v} = A.$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

solution: $\mathbf{x}(t) = e^{At}\mathbf{x}_0$

characterize behavior
through eigenvalues of A

A Few Facts about Eigenvalues and –vectors

The matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ has eigenvalues $\lambda_1 = c + s\mathbf{i}$ $\lambda_2 = c - s\mathbf{i}$

with eigenvectors $u_1 = \begin{bmatrix} 1 \\ -\mathbf{i} \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ +\mathbf{i} \end{bmatrix}$

If c = 0, this is a skew-symmetric matrix

Skew-symmetric matrices: “infinitesimal rotations” (infinitesimal generators of rot.)

For $c = \cos\theta$ and $s = \sin\theta$: 2x2 rotation matrix with

$$\lambda_1 = e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$$

$$\lambda_2 = e^{-\mathbf{i}\theta} = \cos\theta - \mathbf{i}\sin\theta$$

Eigenvalues

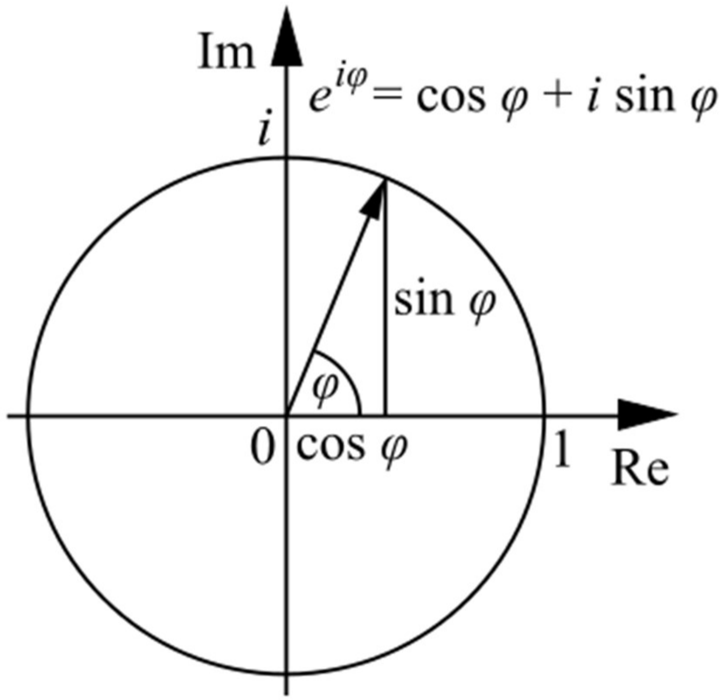
- Symmetric matrix: all eigenvalues are *real*
- Skew-symmetric matrix: all eigenvalues are *pure imaginary*

Euler’s Formula

Can be derived from the infinite power series for exp(), cos(), sin()

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi} + 1 = 0$$



Matrix Exponentials

Defined via same power series as usual exponential

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

Easy if *X* is diagonalizable

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \exp(X) = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

Exponentials of anti-symmetric matrices are rotation matrices

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

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Complex eigenvalues lead to rotation

$$X = \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} \mathrm{Re}(e^{\lambda t}) & -\mathrm{Im}(e^{\lambda t}) \\ \mathrm{Im}(e^{\lambda t}) & \mathrm{Re}(e^{\lambda t}) \end{pmatrix}$$

$$\lambda_{1,2} = a \pm \mathbf{i}\omega$$

Matrix Exponentials

Defined via same power series as usual exponential

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots$$

Easy if *X* is diagonalizable

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Complex eigenvalues lead to rotation

$$X = \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix} \qquad \exp(tX) = e^{at} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}$$

$$\lambda_{1,2} = a \pm \mathbf{i}\omega$$

Classification of Critical Points

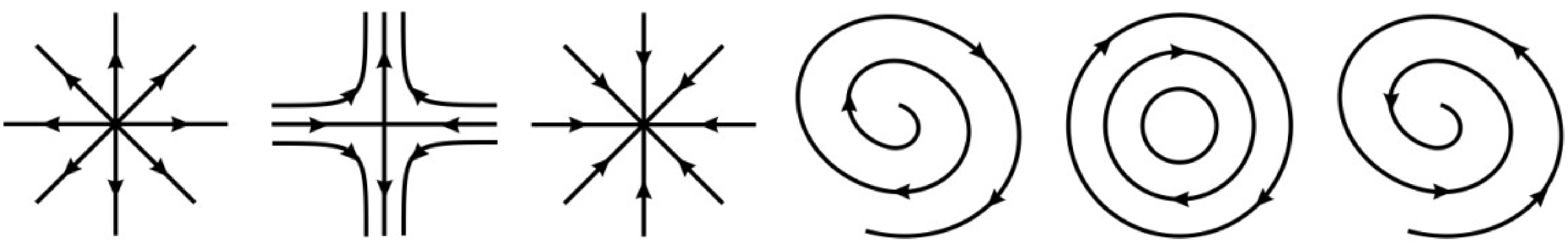
(Isolated) critical point (equilibrium point)

- Velocity vanishes (all components zero)

$\mathbf{v}(\mathbf{x}_c) = \mathbf{0} \quad \text{with} \quad \mathbf{v}(\mathbf{x}_c \pm \boldsymbol{\epsilon}) \neq \mathbf{0}$ det($\nabla\mathbf{v}(\mathbf{x}_c)$) $\neq 0$

Characterize using velocity gradient $\nabla\mathbf{v}$ at critical point $\mathbf{x}_c$

- Look at eigenvalues (and eigenvectors) of $\nabla\mathbf{v}$



Repelling node $R_1, R_2 > 0$, $I_1 = I_2 = 0$

Saddle point $R_1 < 0, R_2 > 0$, $I_1 = I_2 = 0$

Attracting node $R_1, R_2 < 0$, $I_1 = I_2 = 0$

Repelling focus $R_1 = R_2 > 0$, $I_1 = -I_2 \neq 0$

Center $R_1 = R_2 = 0$, $I_1 = -I_2 \neq 0$

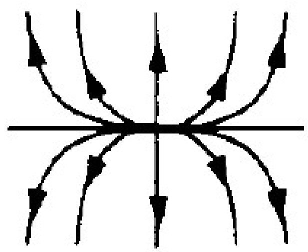
Attracting focus $R_1 = R_2 < 0$, $I_1 = -I_2 \neq 0$

the first three phase portraits are special cases, see later slides!

A Few Details (1)

Repelling/attracting nodes

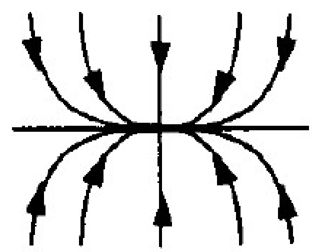
- Do not necessarily imply that streamlines are straight lines (do not confuse with the linear system of ODEs!)
- They are only straight lines when both eigenvalues are real and have the same sign, *and are also equal* (as in the phase portraits before)
- If they are not equal:



Repelling Node

R1, R2 > 0

I1, I2 = 0



Attracting Node

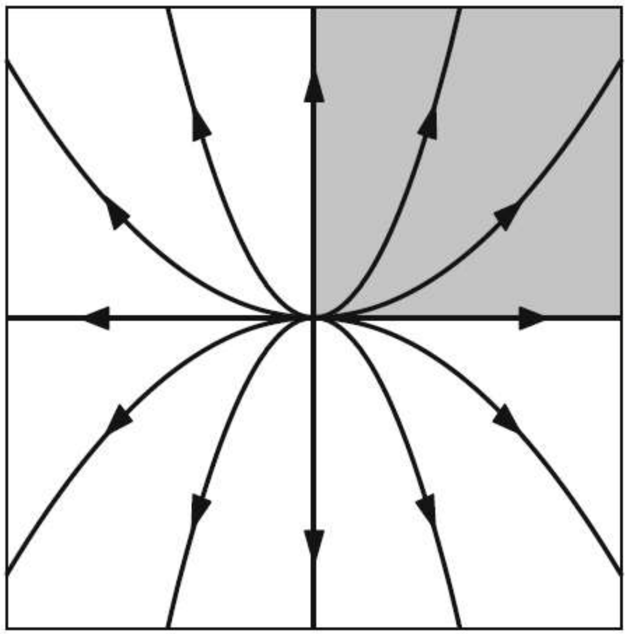
R1, R2 < 0

I1, I2 = 0

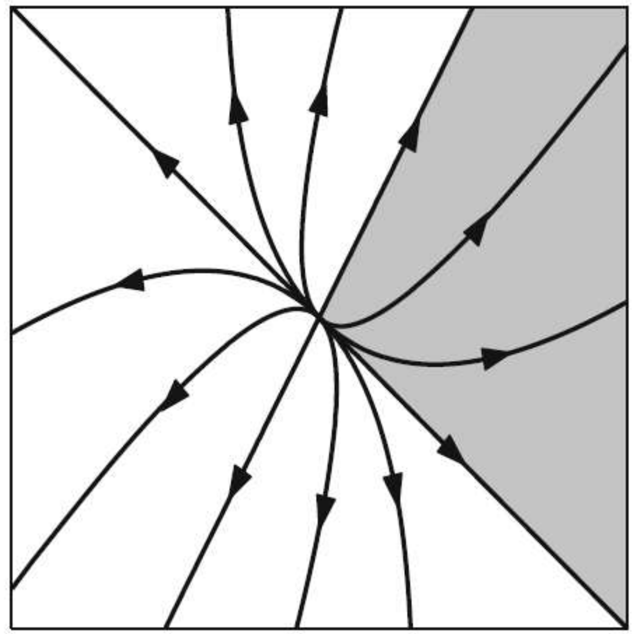
A Few Details (2)

What about skew axes?

- Both of the systems below have eigenvalues 3 and 6
- Jordan normal form (Jordan canonical form) gives details



$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$



$$\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}$$

Jordan Normal Form (2x2 Matrix)

For every real 2x2 matrix A there is an invertible P such that

$P^{-1}AP$ is one of the following Jordan matrices (all entries are real):

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$ (defective matrix)

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Each of these has its corresponding rule for constructing P

- Example on prev. slide (the two eigenvectors are not orthogonal):

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

See also *algebraic* and *geometric multiplicity* of eigenvalues

Another Example

$P^{-1}AP$ has form J_1

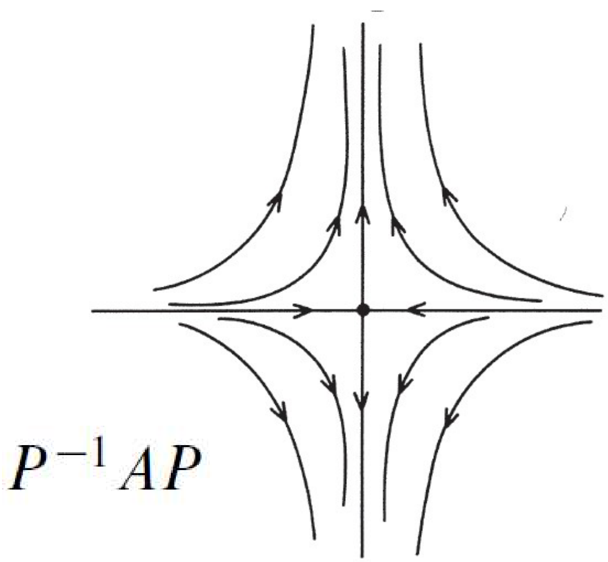
$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

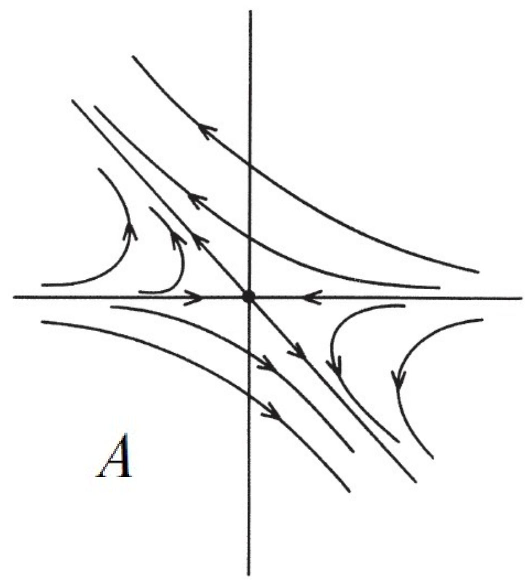
Eigenvalues:

$\lambda_1 = -1$

$\lambda_2 = 2$

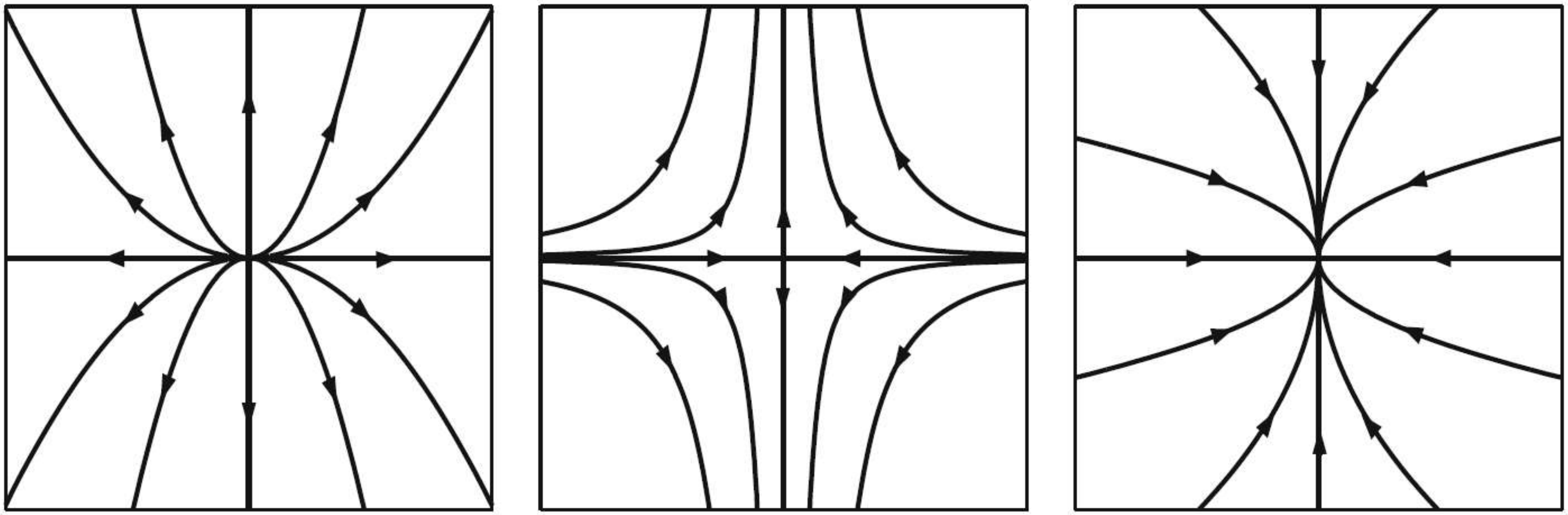
$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$





Jordan Form Characterization (1)

Phase portraits corresponding to Jordan matrix $J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

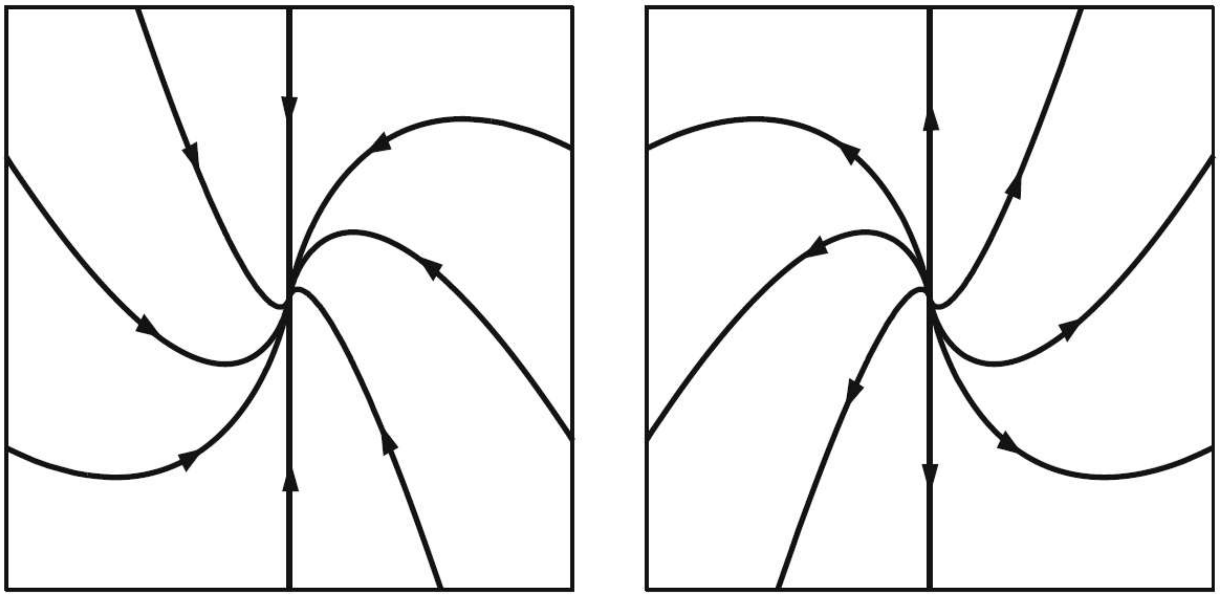


$0 < \lambda_1 < \lambda_2$ unstable node | $\lambda_1 < 0 < \lambda_2$ saddle | $\lambda_1 < \lambda_2 < 0$ stable node

Jordan Form Characterization (2)

Phase portraits corresponding to Jordan matrix

$$J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$



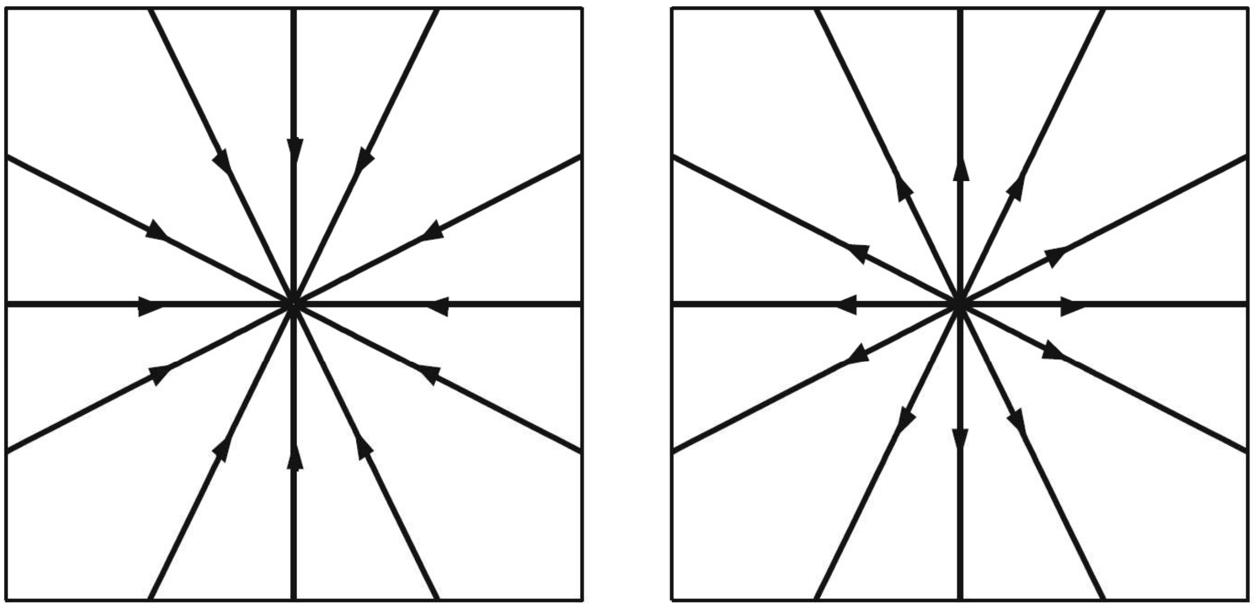
$\lambda < 0$
stable improper node

$\lambda > 0$
unstable improper node

Jordan Form Characterization (3)

Phase portraits corresponding to Jordan matrix

$$J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

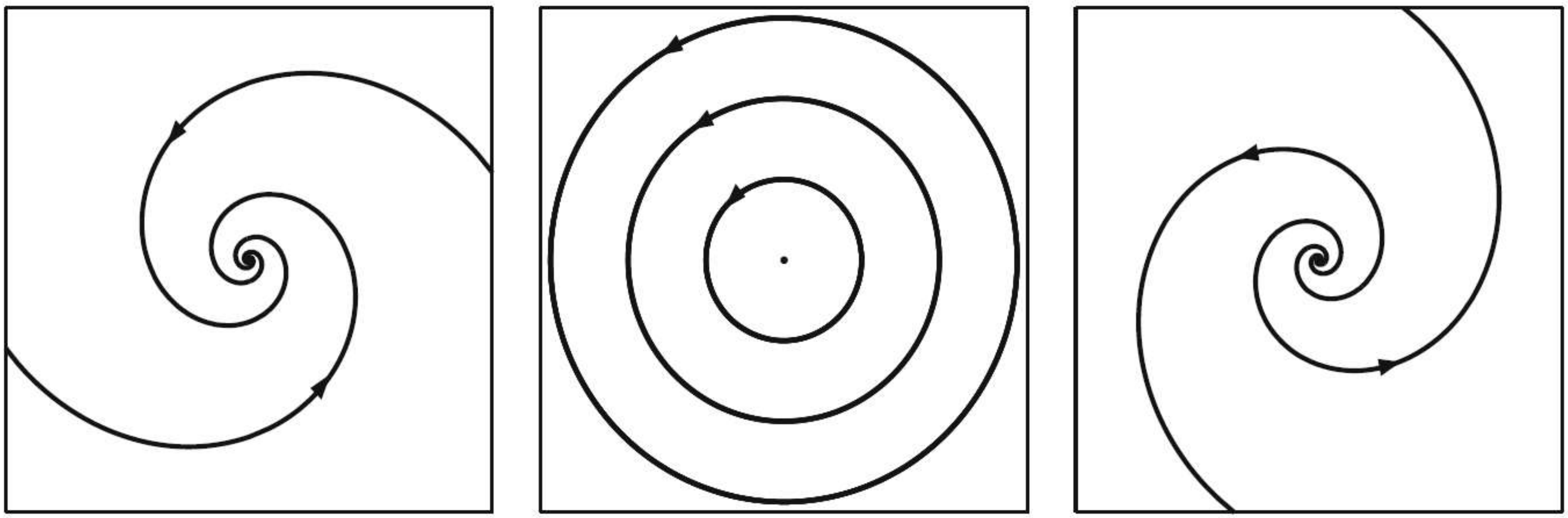


$\lambda < 0$
stable star node

$\lambda > 0$
unstable star node

Jordan Form Characterization (4)

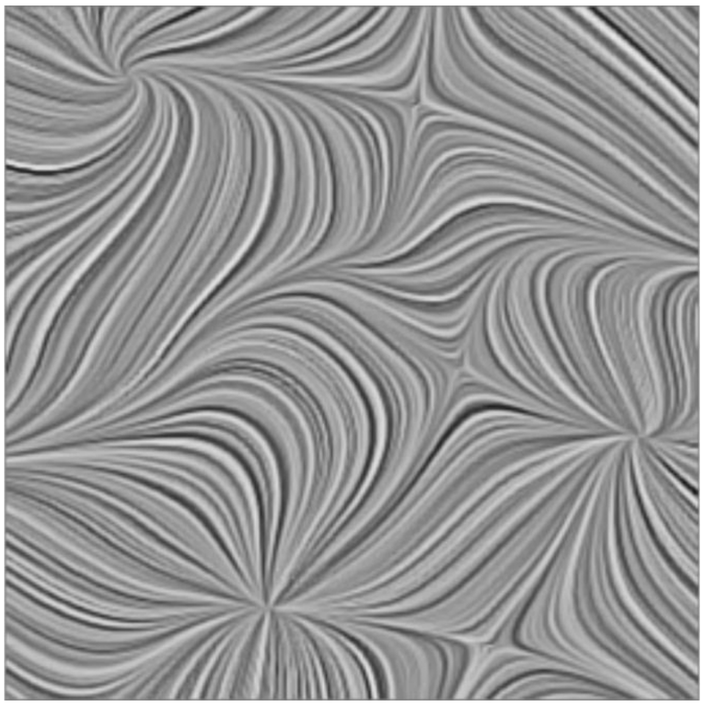
Phase portraits corresponding to Jordan matrix $J_4 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$



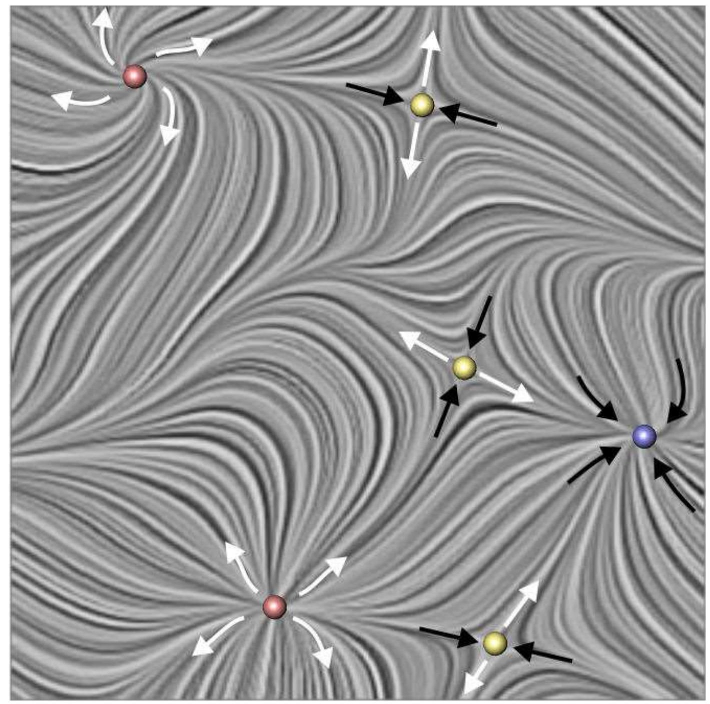
$a < 0$ stable spiral node | $a = 0$ center | $a > 0$ unstable spiral node

Critical Points (Steady Flow!)

Classify critical points according to the *eigenvalues* of the velocity gradient tensor at the critical point

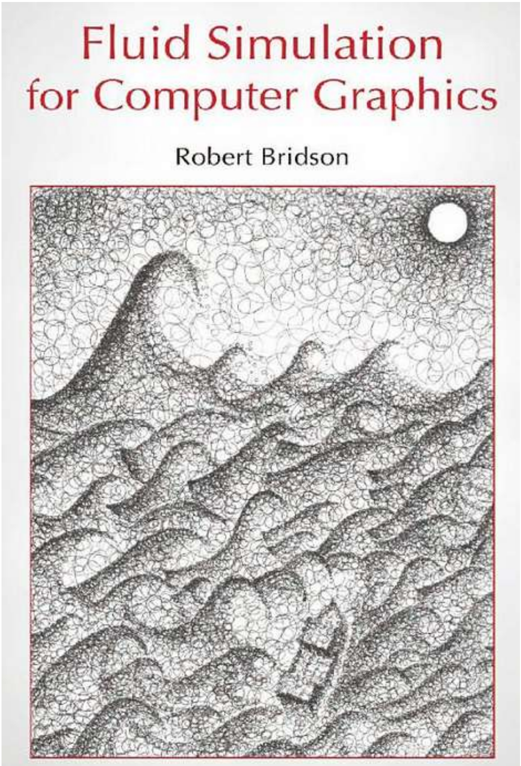


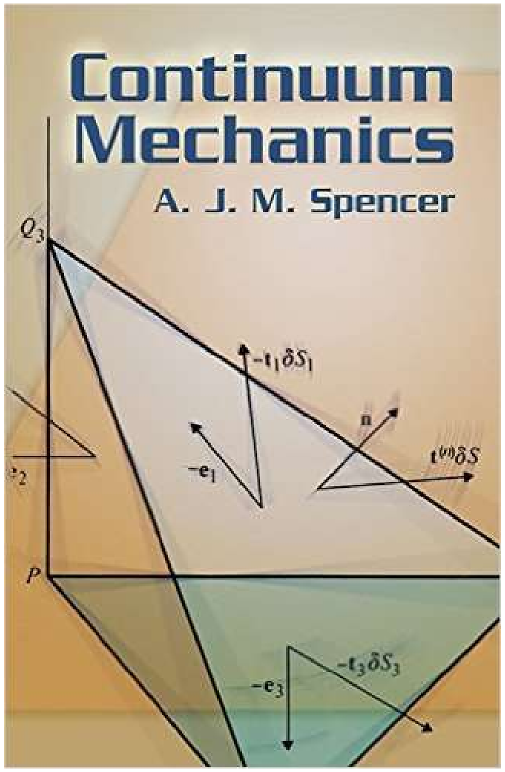
stream lines (LIC)

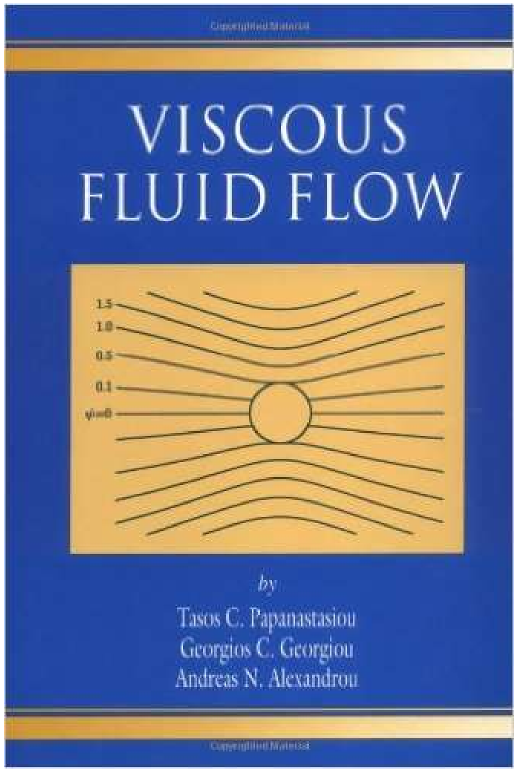


critical points ($\mathbf{v} = 0$)

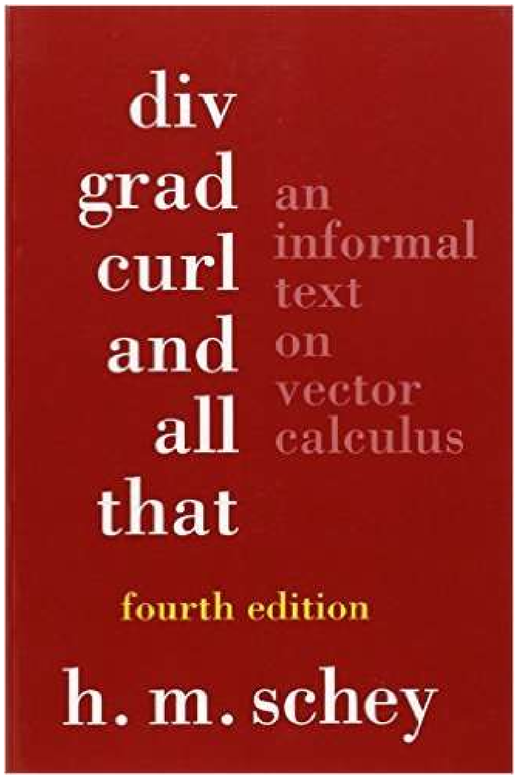
Recommended Books (1)

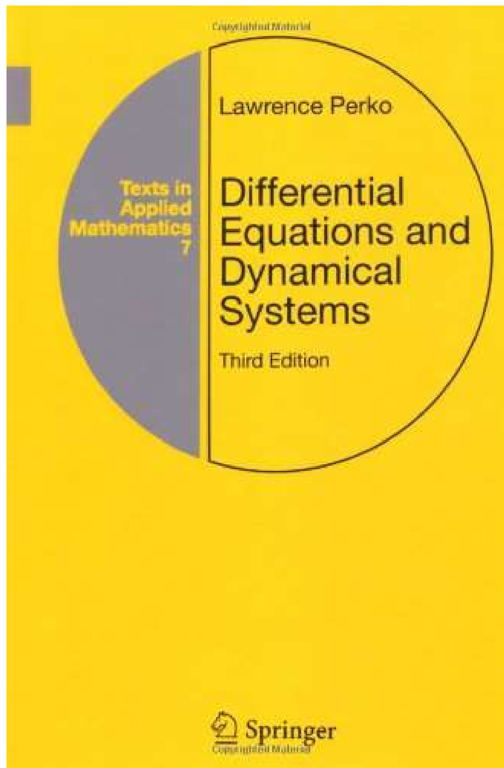


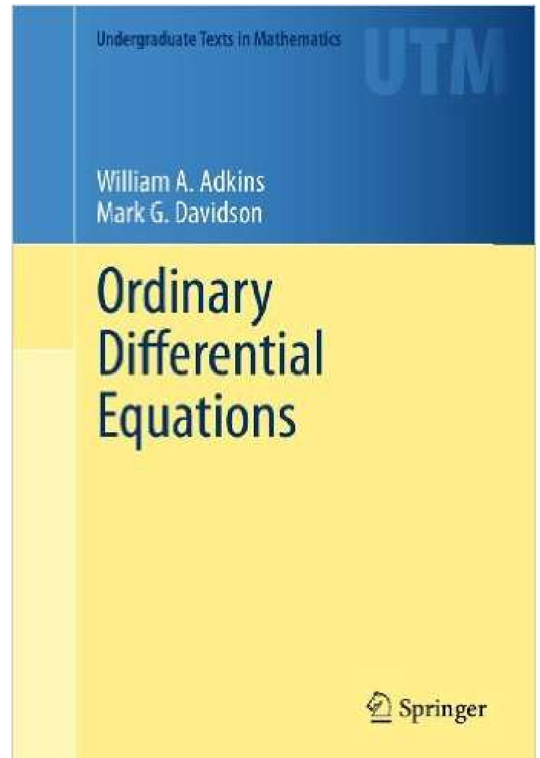




Recommended Books (2)







Thank you.

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