King Abdullah University of Science and Technology

KAUST

CS 247 – Scientific Visualization
Lecture 10: Scalar Fields, Pt. 6

Markus Hadwiger, KAUST

Reading Assignment #5 (until Mar 1)

Read (required):

- Gradients of scalar-valued functions

  `https://en.wikipedia.org/wiki/Gradient`

- Critical points

  `https://en.wikipedia.org/wiki/Critical_point_(mathematics)`

- Multivariable derivatives and differentials

  `https://en.wikipedia.org/wiki/Total_derivative`

  `https://en.wikipedia.org/wiki/Differential_of_a_function#Differentials_in_several_variables`

  `https://en.wikipedia.org/wiki/Hessian_matrix`

- Dot product, inner product (more general)

  `https://en.wikipedia.org/wiki/Dot_product`

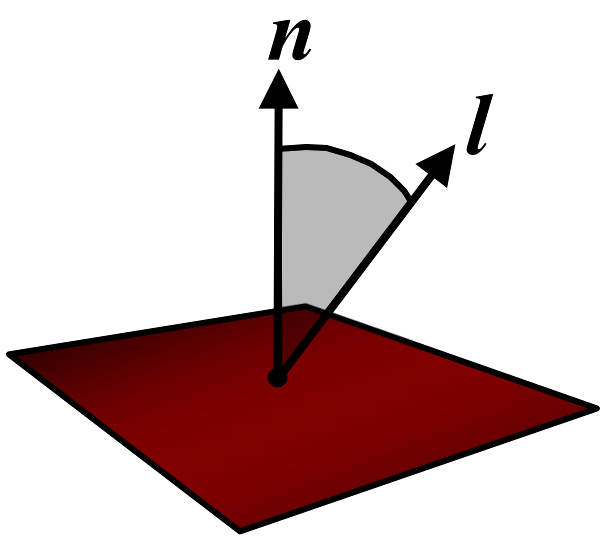
  `https://en.wikipedia.org/wiki/Inner_product_space`

Local Shading Equations

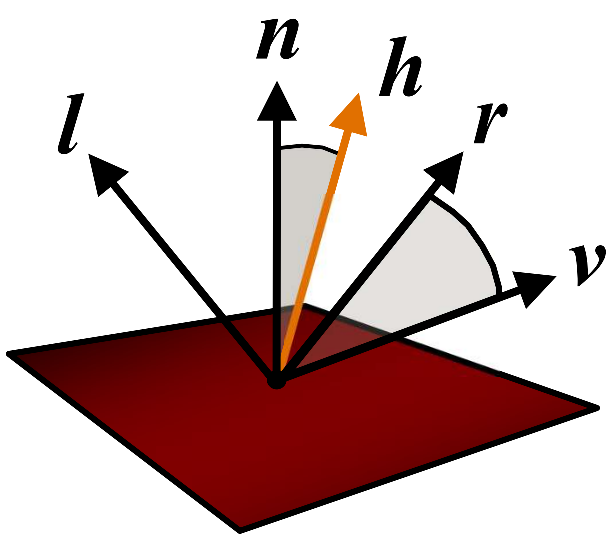
Standard volume shading adapts surface shading

Most commonly Blinn/Phong model

But what about the "surface" normal vector?



diffuse reflection



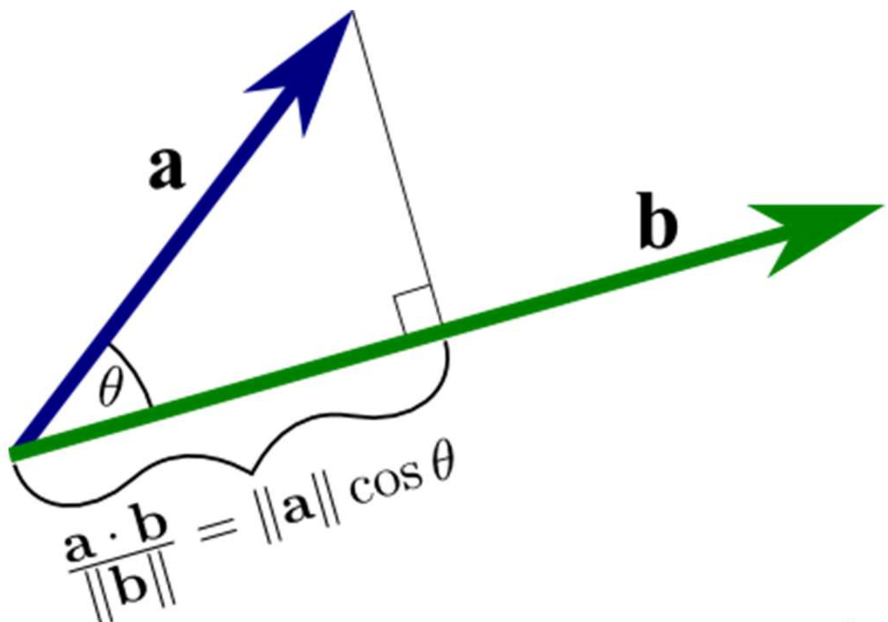
specular reflection

The Dot Product (Scalar / Inner Product)

Cosine of angle between two vectors times their lengths

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i \qquad \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

(standard inner product in Cartesian coordinates)



Many uses:

- Project vector onto another vector,
  project into basis,
  project into tangent plane,
  ...

Gradient and Directional Derivative

Gradient $\nabla f(x,y,z)$ of scalar function $f(x,y,z)$: (in Cartesian coordinates)

$$\nabla f(x,y,z) = \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z} \right)^T$$

Directional derivative in direction $\mathbf{u}$:

$$D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$$

And therefore also:

$$D_{\mathbf{u}} f(x,y,z) = ||\nabla f||\ ||\mathbf{u}||\ \cos\theta$$

Gradient and Directional Derivative

Gradient $\nabla f(x,y,z)$ of scalar function $f\left(x,y,z\right)$: (in Cartesian coordinates)

$$\nabla f(x,y,z) = \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z} \right)^T$$

(Cartesian vector components; basis vectors not shown)

But: always need **basis vectors**! With Cartesian basis:

$$\nabla f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\,\mathbf{i} + \frac{\partial f(x,y,z)}{\partial y}\,\mathbf{j} + \frac{\partial f(x,y,z)}{\partial z}\,\mathbf{k}$$

What about the Basis?

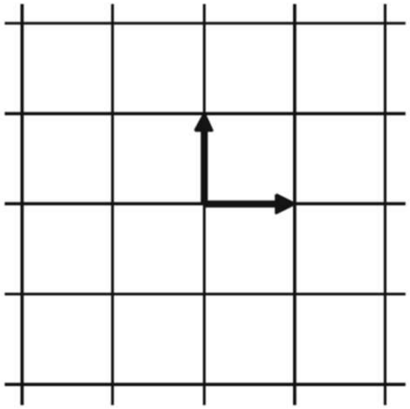
On the previous slide, this actually meant

$$\nabla f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\,\mathbf{i}(x,y,z) + \frac{\partial f(x,y,z)}{\partial y}\,\mathbf{j}(x,y,z) + \frac{\partial f(x,y,z)}{\partial z}\,\mathbf{k}(x,y,z)$$

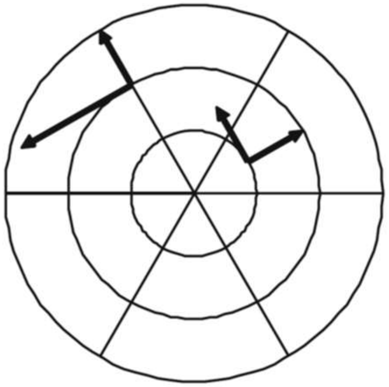
It's just that the Cartesian basis vectors are the same everywhere...

But this is not true for many other coordinate systems:

Cartesian coordinates



polar coordinates



The Gradient as Normal Vector

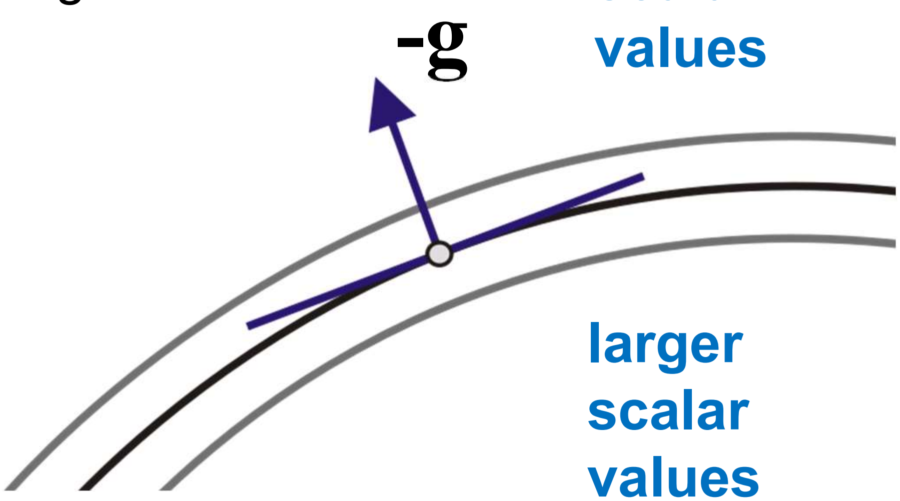
Gradient of the scalar field gives direction+magnitude of fastest change

$$\mathbf{g} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)^{\mathbf{T}}$$

Local approximation to isosurface at any point:
tangent plane = plane orthogonal to gradient

Normal of this isosurface:
normalized gradient vector
(negation is common convention)

$$\mathbf{n} = -\mathbf{g}/|\mathbf{g}|$$



The Gradient as a Differential Form

The gradient as a *differential* (differential 1-form) is the "primary" concept

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

A differential 1-form is a scalar-valued linear function that takes a (direction) vector as input, and gives a scalar as output

Each of the 1-forms df, dx, dy, dz takes a (direction) vector as input, gives scalar as output

In the expression of the gradient df above, all 1-forms on the right-hand side get the same vector as input

df is simply a linear combination of the coordinate differentials dx, dy, dz

The Gradient as a Differential Form

The gradient as a *differential* (differential 1-form) is the “primary” concept

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

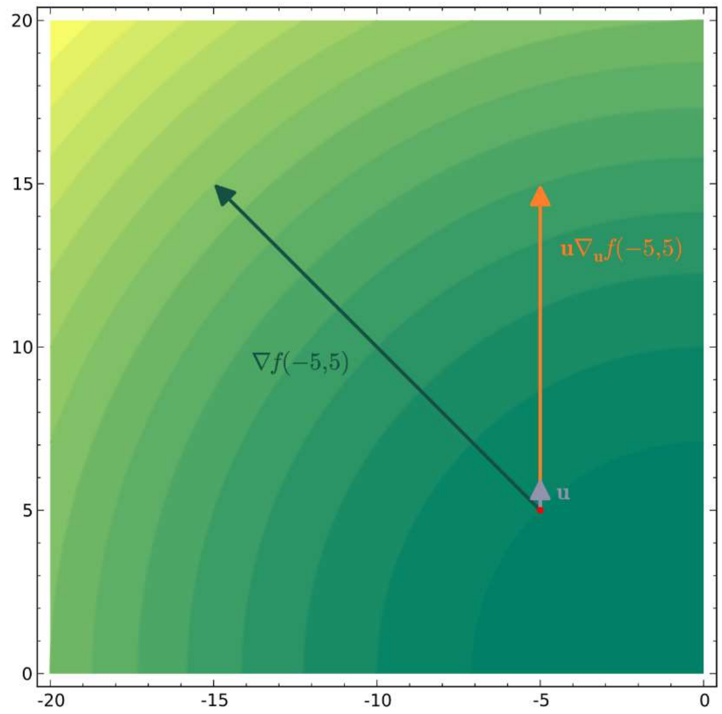
The directional derivative and the gradient vector

$$D_{\mathbf{u}} f = df(\mathbf{u})$$
$$df(\mathbf{u}) = \nabla f \cdot \mathbf{u}$$

The gradient vector is then *defined*, such that:

$$\nabla f \cdot \mathbf{u} := df(\mathbf{u})$$

Gradient Vectors and Differential 1-Forms

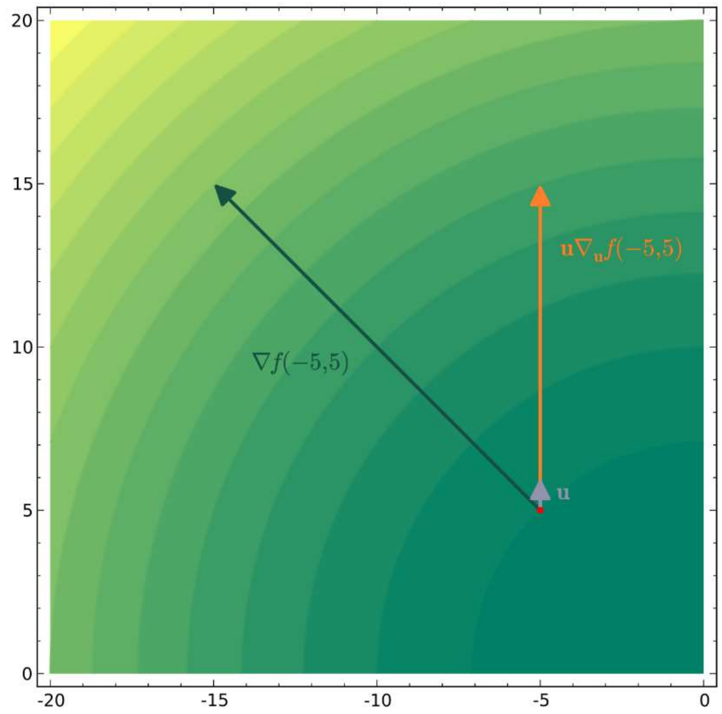


from Wikipedia (for **u** a unit vector),

the function here is $f(x,y) = x^2 + y^2$

$$\nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$$

Gradient Vectors and Differential 1-Forms



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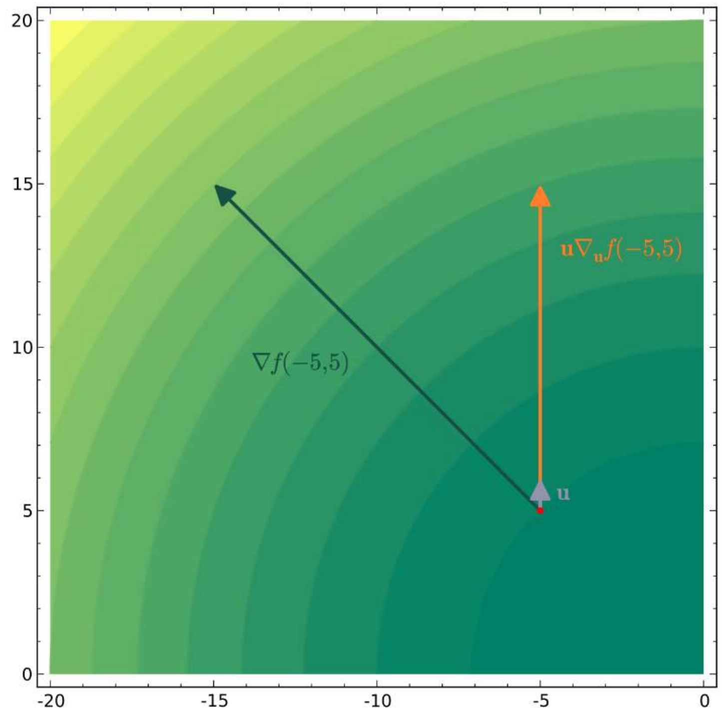
the function here is $f(x,y) = x^2 + y^2$

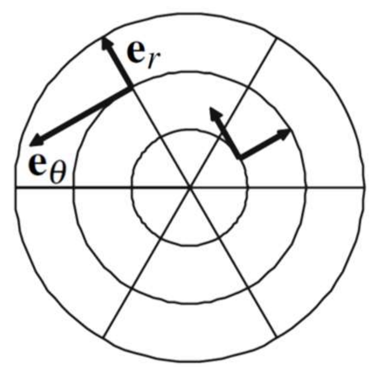
$$\nabla f(x,y) = 2x\,\mathbf{e}_x + 2y\,\mathbf{e}_y$$

$$df(x,y) = 2x\,dx + 2y\,dy$$

Gradient Vectors and Differential 1-Forms



how about in polar coordinates?



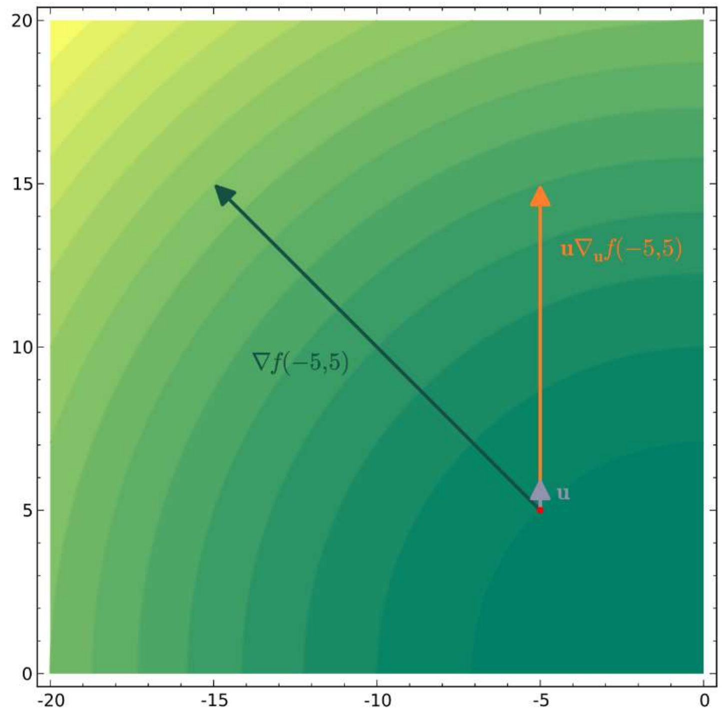
from Wikipedia (for **u** a unit vector),

the function here is $f(r,\theta) = r^2$

$$\nabla f(r,\theta) = 2r\,\mathbf{e}_r + 0\frac{1}{r^2}\,\mathbf{e}_\theta = 2r\,\mathbf{e}_r$$

$$df(r,\theta) = 2r\,dr + 0\,d\theta = 2r\,dr$$

Gradient Vectors and Differential 1-Forms



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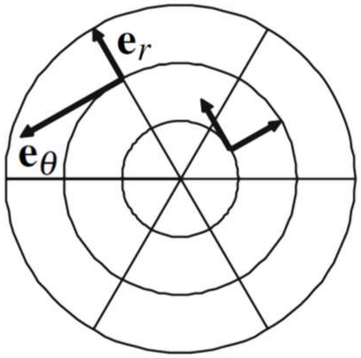
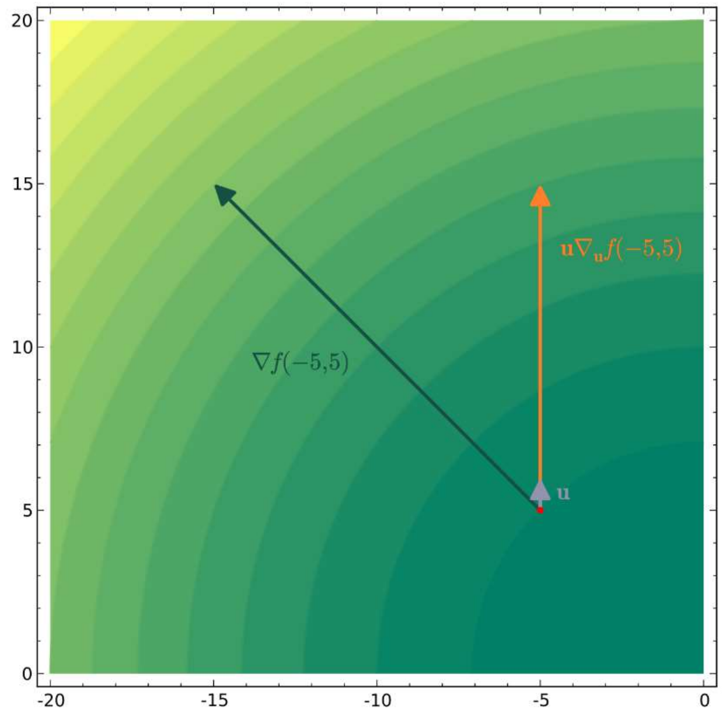


Gradient Vectors and Differential 1-Forms



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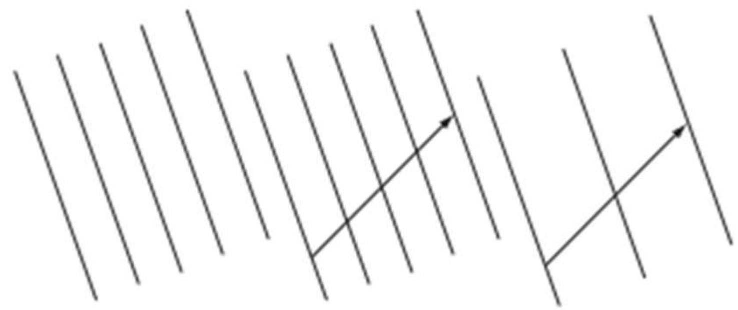
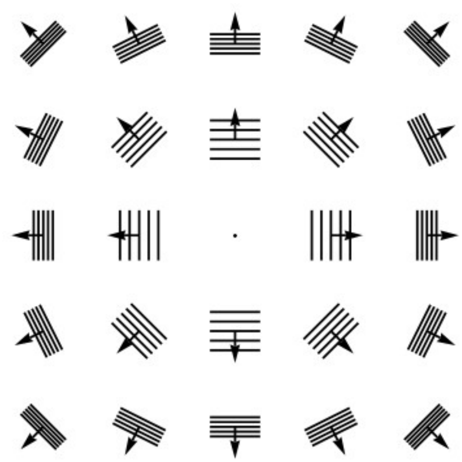
different 1-forms
evaluated in some direction



1-form (field) df



Interlude: Tensor Calculus

In tensor calculus, first-order tensors can be

- Contravariant $\mathbf{v} = v^i \, \mathbf{e}_i$
- Covariant $\boldsymbol{\omega} = v_i \, \boldsymbol{\omega}^i$

The gradient vector is a contravariant vector $\mathbf{v} = v^i \, \boldsymbol{\partial}_i$

The gradient 1-form is a covariant vector (a covector) $df = \frac{\partial f}{\partial x^i} \, dx^i$

Very powerful; necessary for non-Cartesian coordinate systems

On (intrinsically) curved manifolds (sphere, ...):
Cartesian coordinates not even possible

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This is also the fundamental reason why in graphics a normal vector transforms differently: as a covector, not as a vector!

(typical graphics rule: **n** transforms with transpose of inverse matrix)

Thank you.

Thanks for material

- Helwig Hauser
- Eduard Gröller
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- Philipp Muigg
- Christof Rezk-Salama