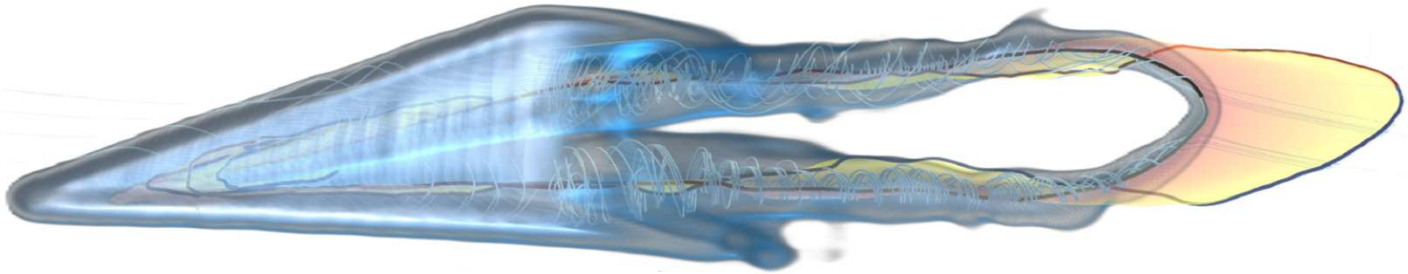
Lagrangian and Hamiltonian Phase Spaces for Reference Frames and Vortices in Unsteady Flow

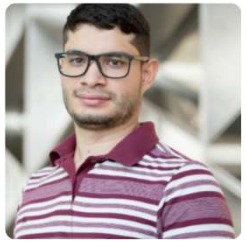


جامعة الملك عبدالله
للعلوم والتقنية
King Abdullah University of
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Markus Hadwiger
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Peter Rautek


Markus Hadwiger

Alumni













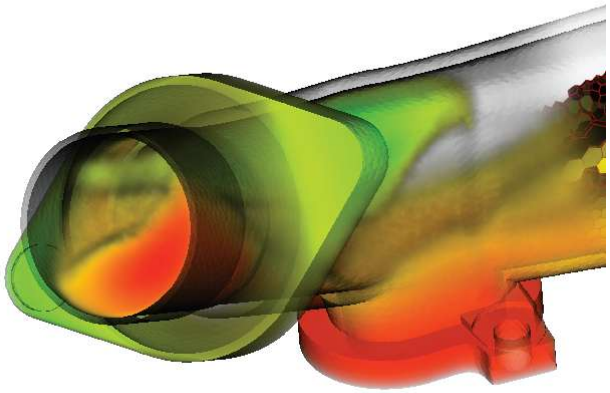


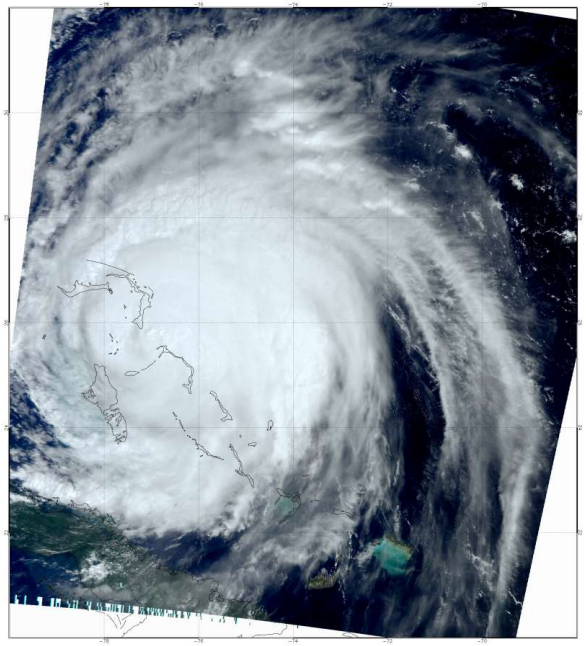

Flow Visualization and Reference Frames

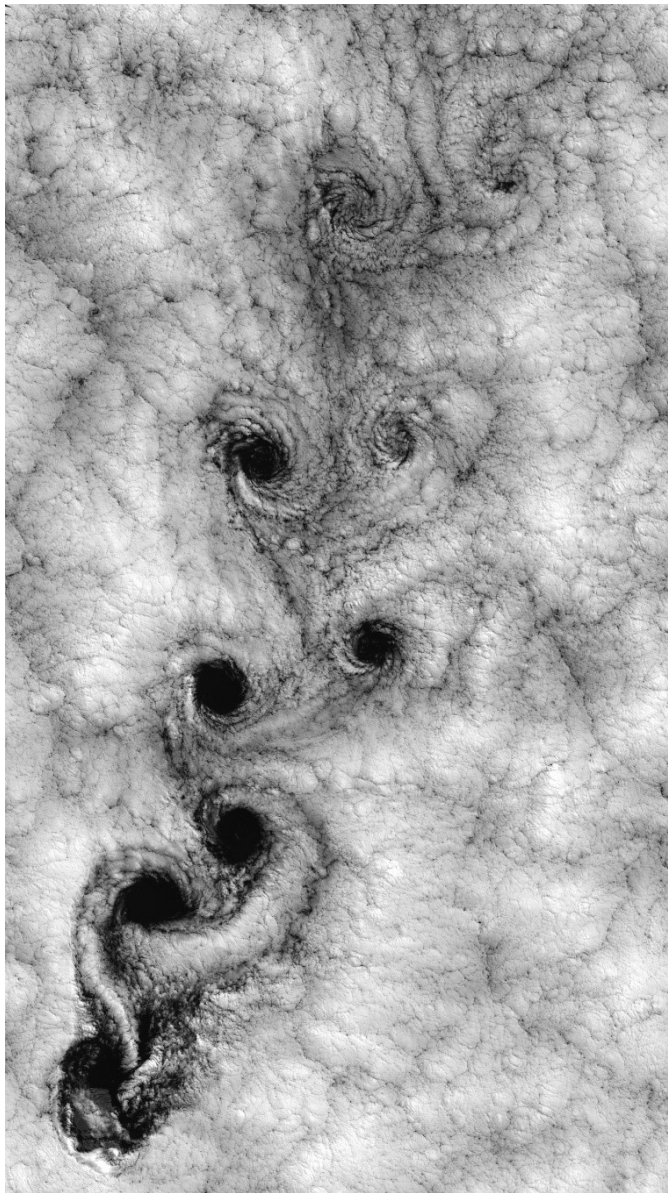
Flow Phenomena

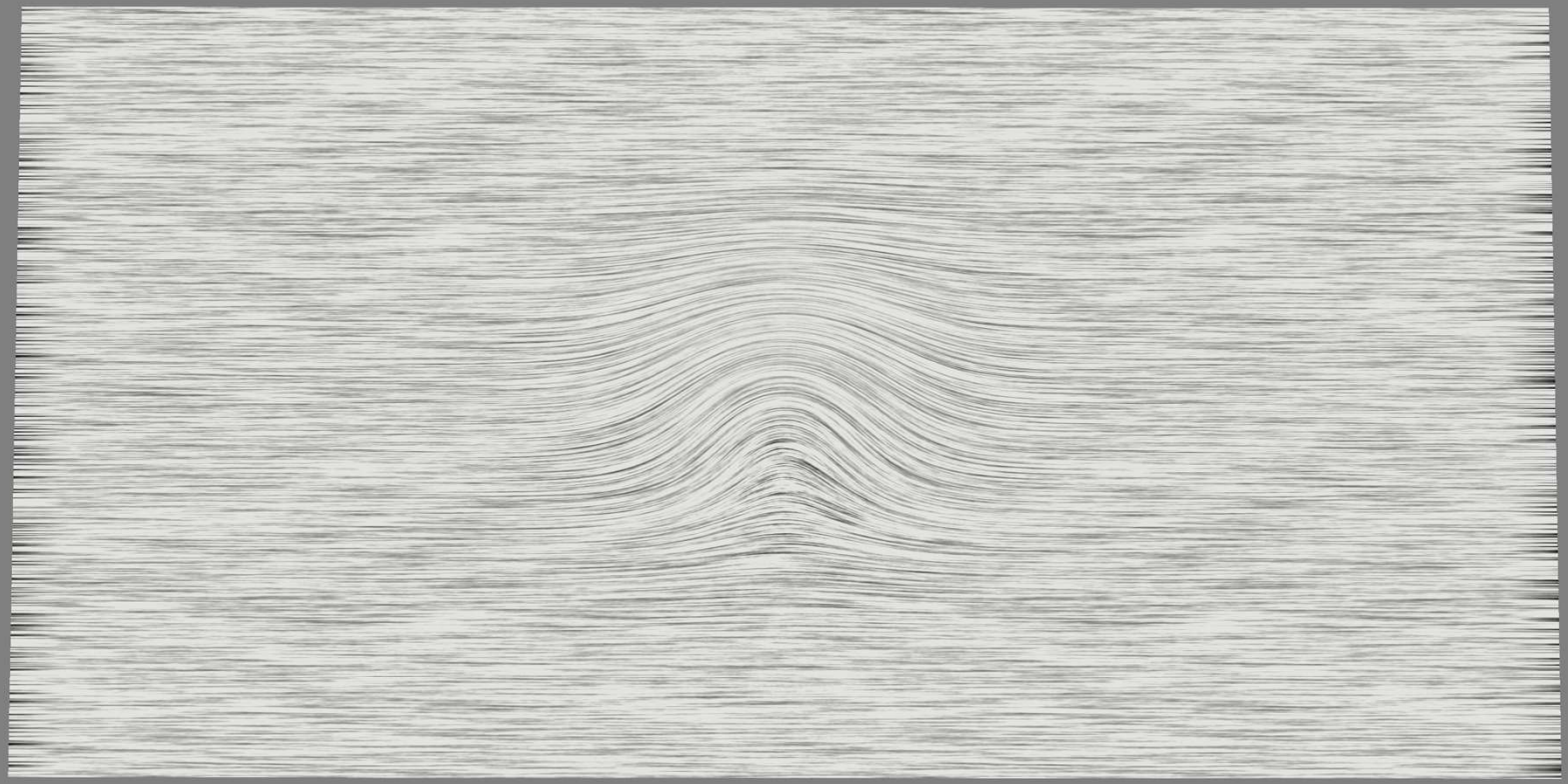
Vector fields: fluid velocity, force fields, ...

- CFD, hurricanes, oceanography, ...





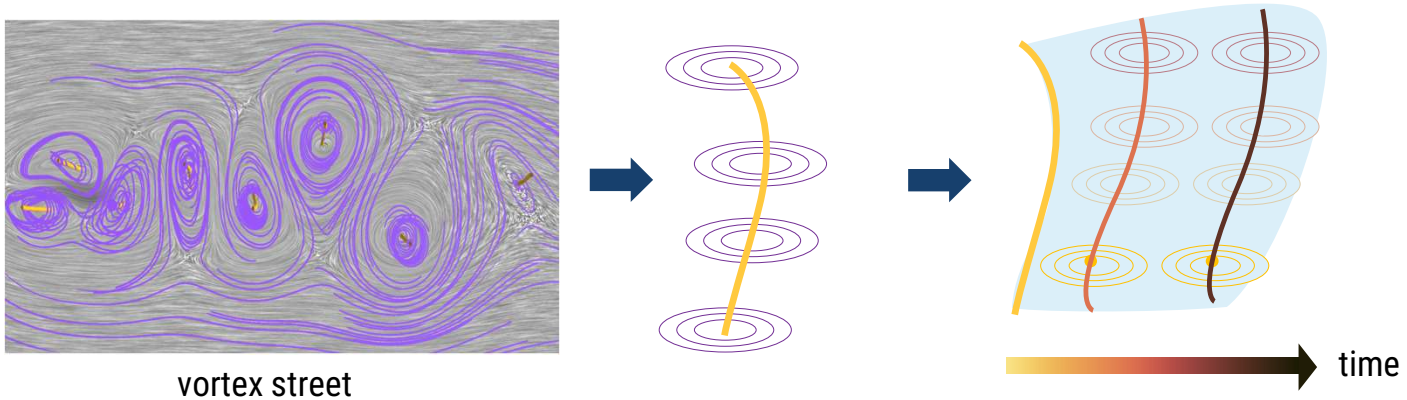




Vortices and Optimal Reference Frames (1)

No universal agreement on what constitutes a vortex

- In a reference frame following a vortex, streamlines should be circles/spirals
- Reference frame cannot follow multiple vortices, or in 3D even a single vortex



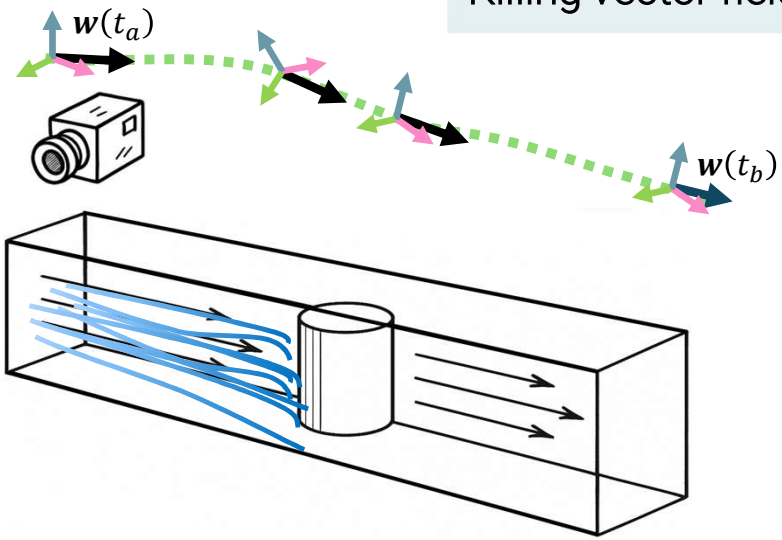
vortex street

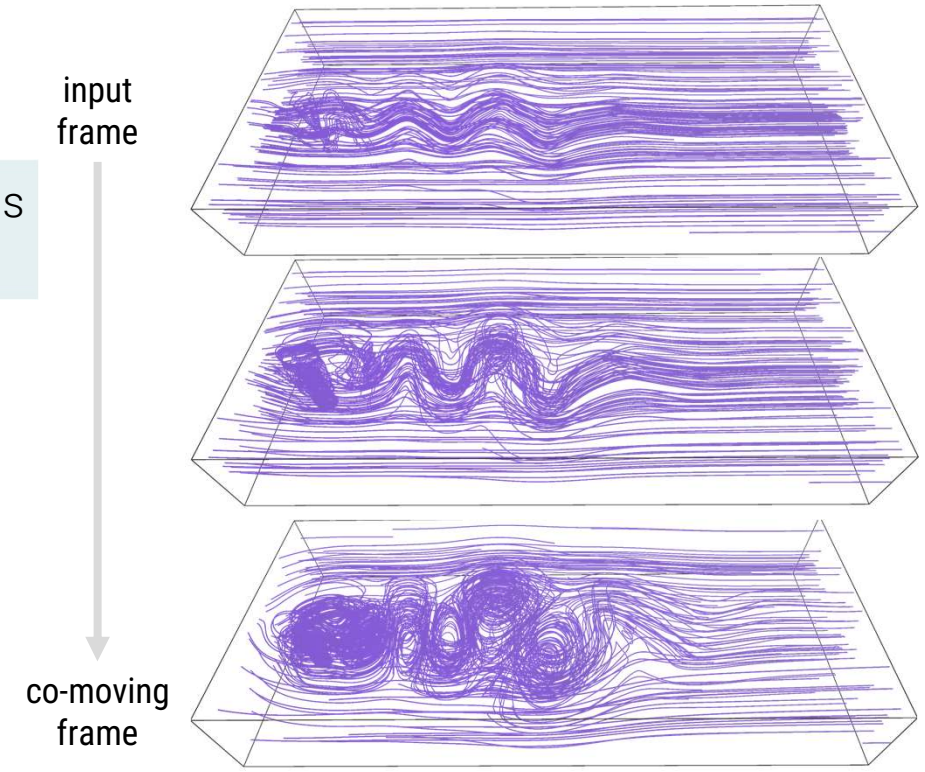
Vortices and Optimal Reference Frames (2)

Observer motion is rigid transformation

$$\boldsymbol{x}^* = \boldsymbol{Q}(t)\,\boldsymbol{x} + \boldsymbol{c}(t)$$

Observer velocity field $\boldsymbol{w}(t)$ is Killing vector field





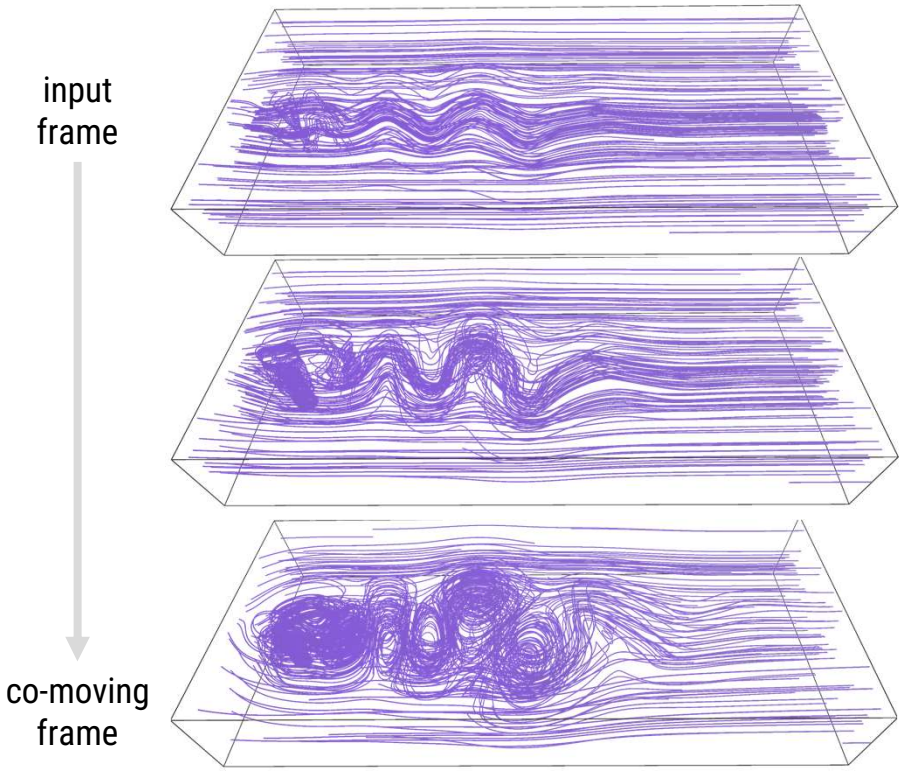
Vortices and Optimal Reference Frames (3)

Optimal observer computation

Compute observer field $\boldsymbol{w}(t)$ such that the *observed time derivative* is minimized

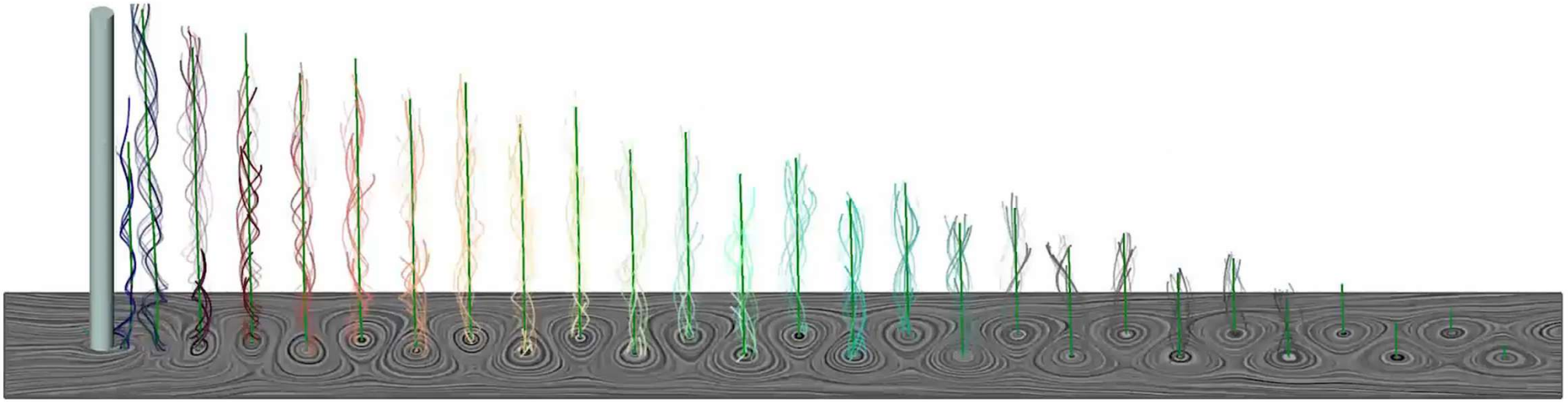
$$\frac{D}{Dt}(\boldsymbol{v}-\boldsymbol{w}) = \left(\frac{\partial}{\partial t} + \mathcal{L}_{\boldsymbol{w}}\right)(\boldsymbol{v}-\boldsymbol{w})$$

observed time derivative
(specific time-dependent Lie derivative)



Vortices and Optimal Reference Frames (4)

Instead of a single reference frame given by a Killing field,
we can use an *approximate* Killing field: infinitely many, locally adapted observers



Observer Lie Algebra

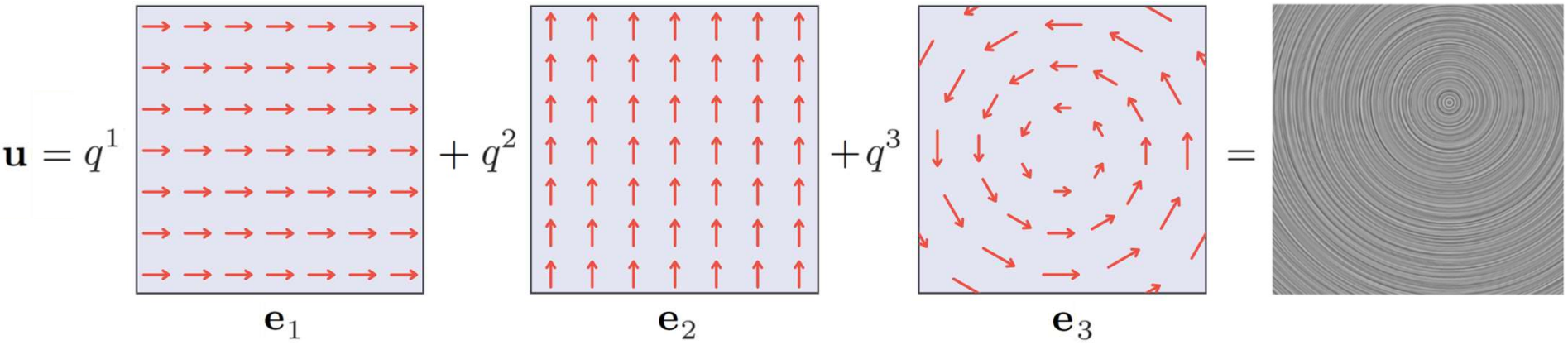
Lie Algebra of Observer Motions (1)



Wilhelm Killing
(1847-1923)

Vector space ("vectors" are vector fields)

$$\mathbf{u}(x,t) = \sum_{i=1}^{k} q^i(t)\,\mathbf{e}_i(x,t)$$ linear combination of vector field basis



Lie Algebra of Observer Motions (1)

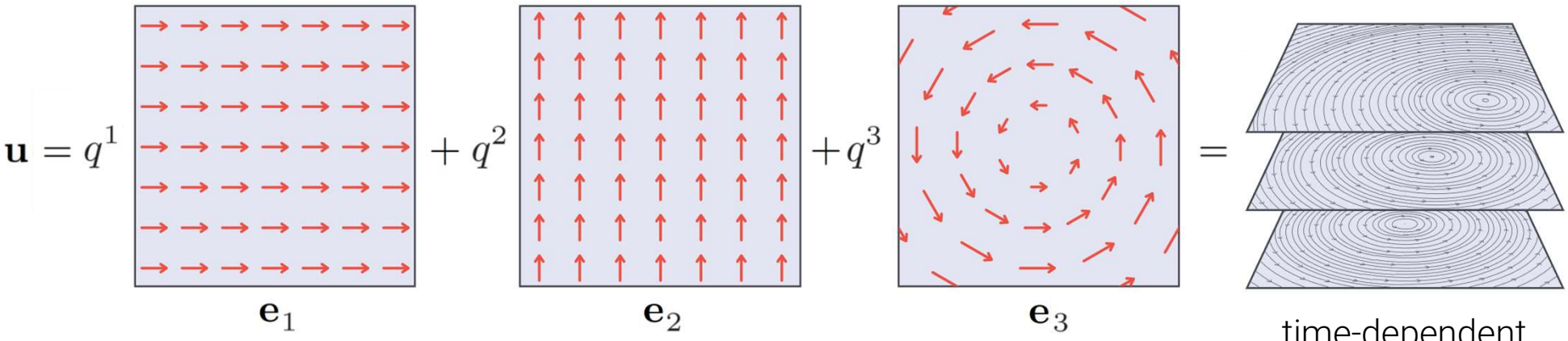


Wilhelm Killing
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Vector space (“vectors” are vector fields)

$$\mathbf{u}(x,t) = \sum_{i=1}^{k} q^i(t)\,\mathbf{e}_i(x,t)$$

linear combination of vector field basis



time-dependent

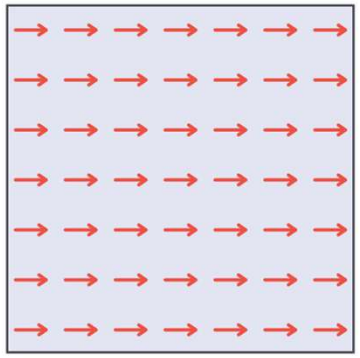
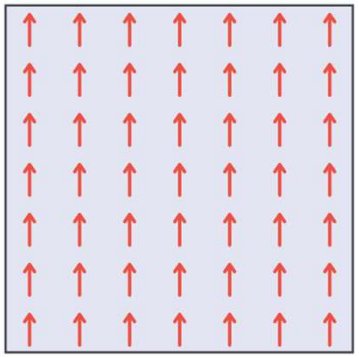
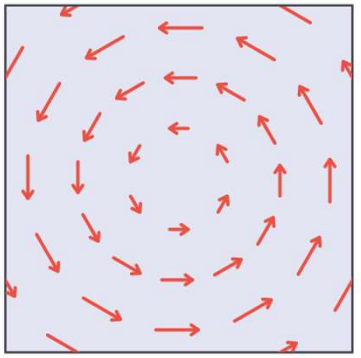
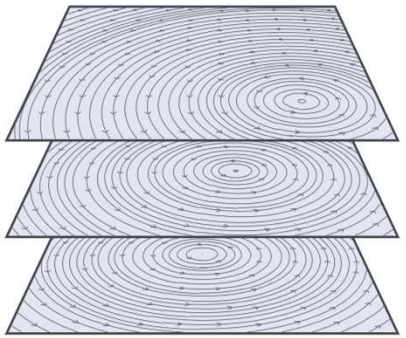
Lie Algebra of Observer Motions (2)



Wilhelm Killing
(1847-1923)

Vector space ("vectors" are vector fields)

$$\mathbf{u}(x,t) = \sum_{i=1}^{k} q^i(t)\,\mathbf{e}_i(x,t)$$ linear combination of vector field basis

$$\frac{\partial}{\partial t}\mathbf{u} = \dot{q}^1$$  $$+\dot{q}^2$$  $$+\dot{q}^3$$  $$=$$ 

$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$

time-dependent

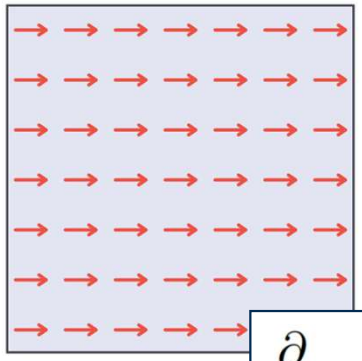
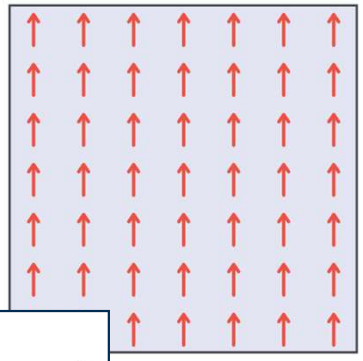
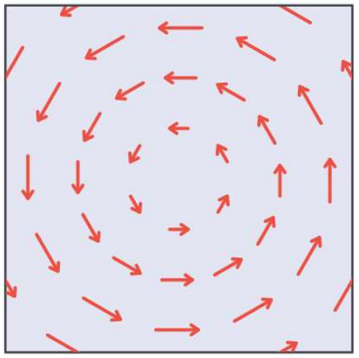
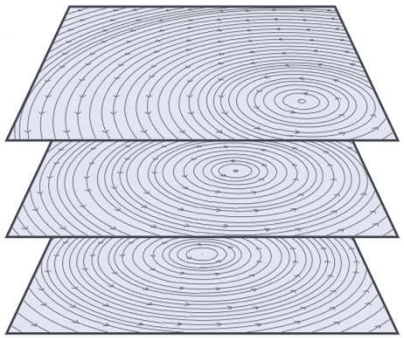
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$$\frac{\partial}{\partial t}\mathbf{u} = \dot{q}^1$$  $$+\dot{q}^2$$  $$+\dot{q}^3$$  $$=$$ 

$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$

$$\frac{\partial}{\partial t}\mathbf{e}_i(x,t) = 0$$

time-dependent

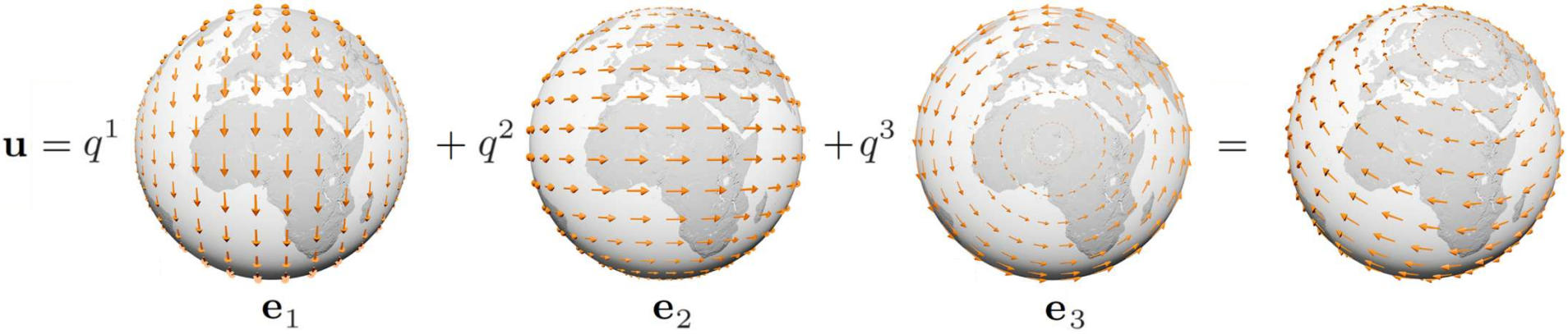
Lie Algebra of Observer Motions (3)



Wilhelm Killing
(1847-1923)

Vector space (“vectors” are vector fields)

$$\mathbf{u}(x,t) = \sum_{i=1}^{k} q^i(t)\,\mathbf{e}_i(x,t)$$ linear combination of vector field basis



Lie Algebra of Observer Motions (3)

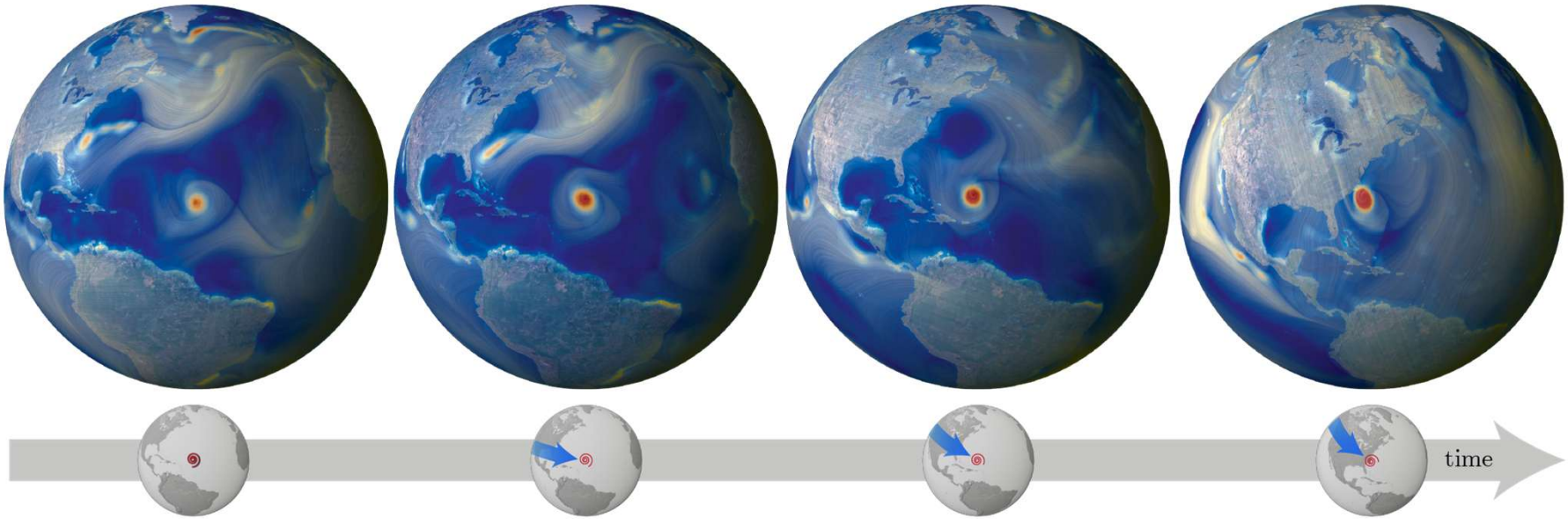


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(1847-1923)

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linear combination of vector field basis



Observer Optimization Methods

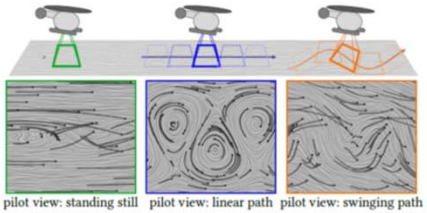
Observer Optimization Methods

Generic Objective Vortices for Flow Visualization

2017

TOBIAS GÜNTHER and MARKUS GROSS, ETH Zürich
HOLGER THEISEL, University of Magdeburg

In flow visualization, vortex extraction is a long-standing and unsolved problem. For decades, scientists developed numerous definitions that characterize vortex regions and their corelines in different ways, but none emerged as ultimate solution. One reason is that almost all techniques have a fundamental weakness: they are not invariant under changes of the reference frame, i.e., they are not objective. This has two severe implications: First, the result depends on the movement of the observer, and second, they cannot track vortices that are moving on arbitrary paths, which limits their reliability and usefulness in practice. Objective measures are rare, but recently gained more attention in the literature. Instead of only introducing a new objective measure, we show in this paper how all existing measures that are based on velocity and its derivatives can be made objective. We achieve this by observing the vector field in optimal local reference frames, in which the temporal derivative of the flow vanishes, i.e., reference frames in which the flow appears steady. The central contribution of our paper is to show that these optimal local reference frames can be found by a simple and elegant linear optimization. We prove that in the optimal frame, all local vortex extraction methods that are based on velocity and its derivatives become objective. We demonstrate our approach with objective counterparts to λ_2, vorticity and Sujudi-Haimes.



pilot view: standing still pilot view: linear path pilot view: swinging path

Fig. 1. Most flow visualization techniques depend on the movement of the observer, i.e., they are not objective. Here, a line integral convolution (time slice) and pathlines (black) are shown for three different reference frame movements: standing still, linearly translating and swinging along a sine curve–unfortunately, all give different results. We solve this ambiguity by finding an optimal local reference frame, in which the flow appears steady. In our optimal frame, all vortex extraction methods that are based on velocity and Jacobian become objective, including vorticity, λ_2 and Sujudi-Haimes.

Observer Optimization Methods

Generic Objective Vortices for Flow Visualization

2017

2018

Time-Dependent Flow seen through Approximate Observer Killing Fields

Markus Hadwiger, Matej Mlejnek, Thomas Theußl, and Peter Rautek

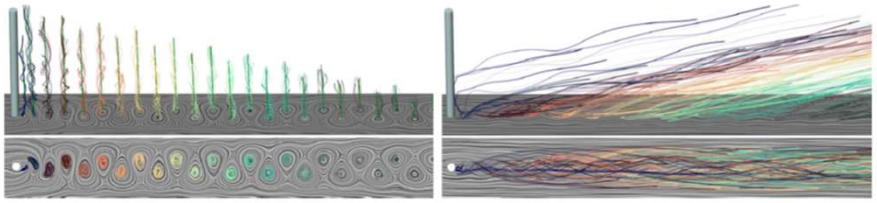
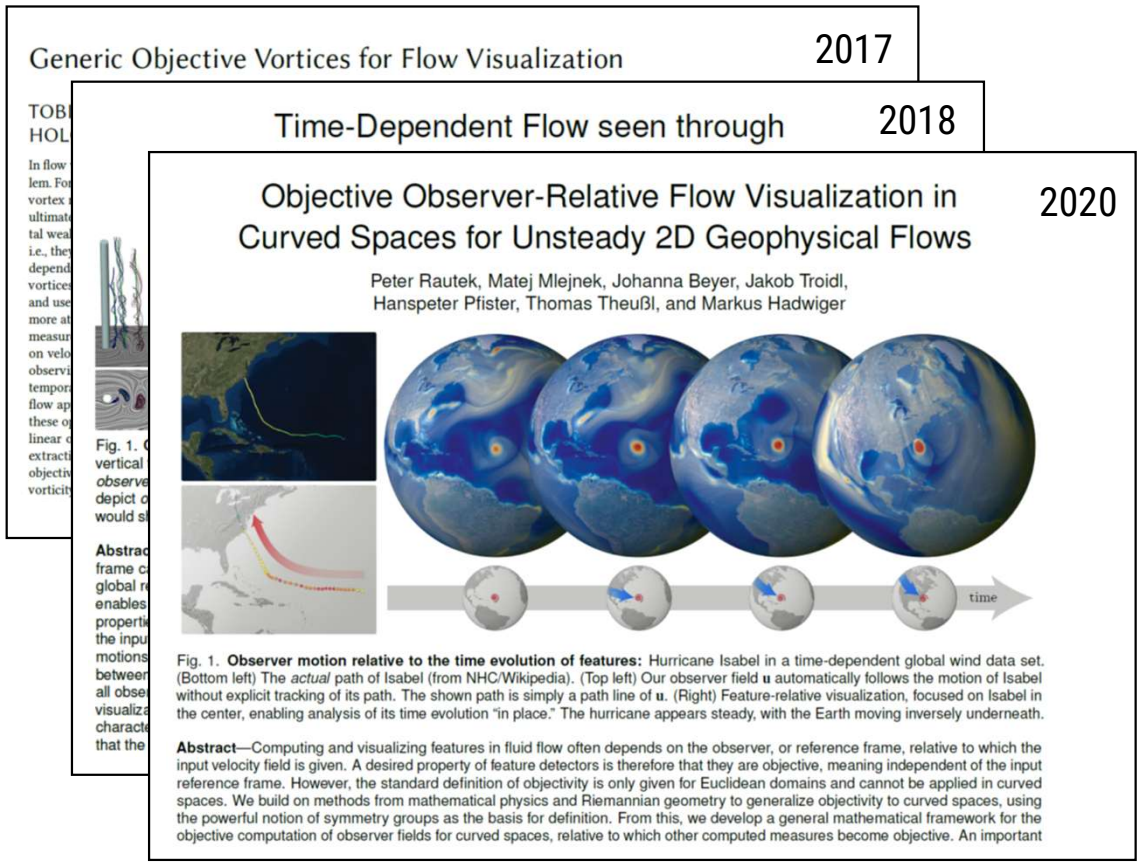


Fig. 1. **Observer field-relative visualization.** 2D von Kármán vortex street behind a circular cylinder. (Top) Space-time views with vertical time axes. (Bottom) Spatial domain. (Left) A *continuous observer field* follows the motion of the input flow by minimizing the *observed time derivative*. *Observed path lines* swirl around "vertical" vortex cores computed in the observed field, while LIC images depict *observed stream lines*. The observer field *jointly* perceives all vortices as stationary. Prior objective vortex detection methods would show the core lines in motion. (Right) The original lab frame observer perceives vortices and path lines as moving rightward.

Abstract—Flow fields are usually visualized relative to a global observer, i.e., a single frame of reference. However, often no global frame can depict all flow features equally well. Likewise, objective criteria for detecting features such as vortices often use either a global reference frame, or compute a separate frame for each point in space and time. We propose the first general framework that enables choosing a smooth trade-off between these two extremes. Using global optimization to minimize specific differential geometric properties, we compute a time-dependent observer velocity field that describes the motion of a continuous field of observers adapted to the input flow. This requires developing the novel notion of an observed time derivative. While individual observers are restricted to rigid motions, overall we compute an approximate Killing field, corresponding to almost-rigid motion. This enables continuous transitions between different observers. Instead of focusing only on flow features, we furthermore develop a novel general notion of visualizing how all observers jointly perceive the input field. This in fact requires introducing the concept of an observation time, with respect to which a visualization is computed. We develop the corresponding notions of observed stream, path, streak, and time lines. For efficiency, these characteristic curves can be computed using standard approaches, by first transforming the input field accordingly. Finally, we prove that the input flow perceived by the observer field is objective. This makes derived flow features, such as vortices, objective as well.

Observer Optimization Methods



Generic Objective Vortices for Flow Visualization 2017

Time-Dependent Flow seen through 2018

Objective Observer-Relative Flow Visualization in Curved Spaces for Unsteady 2D Geophysical Flows 2020

Peter Rautek, Matej Mlejnek, Johanna Beyer, Jakob Troidl, Hanspeter Pfister, Thomas Theußl, and Markus Hadwiger

Fig. 1. **Observer motion relative to the time evolution of features:** Hurricane Isabel in a time-dependent global wind data set. (Bottom left) The *actual* path of Isabel (from NHC/Wikipedia). (Top left) Our observer field **u** automatically follows the motion of Isabel without explicit tracking of its path. The shown path is simply a path line of **u**. (Right) Feature-relative visualization, focused on Isabel in the center, enabling analysis of its time evolution "in place." The hurricane appears steady, with the Earth moving inversely underneath.

Abstract—Computing and visualizing features in fluid flow often depends on the observer, or reference frame, relative to which the input velocity field is given. A desired property of feature detectors is therefore that they are objective, meaning independent of the input reference frame. However, the standard definition of objectivity is only given for Euclidean domains and cannot be applied in curved spaces. We build on methods from mathematical physics and Riemannian geometry to generalize objectivity to curved spaces, using the powerful notion of symmetry groups as the basis for definition. From this, we develop a general mathematical framework for the objective computation of observer fields for curved spaces, relative to which other computed measures become objective. An important

Observer Optimization Methods

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no time derivative minimization

The objective deformation component of a velocity field 2022

Bálint Kaszás [a,*], Tiemo Pedergnana [b], George Haller [a]

[a] Institute for Mechanical Systems, ETH Zurich, Zurich, Switzerland
[b] Institute for Energy and Process Engineering, ETH Zurich, Zurich, Switzerland

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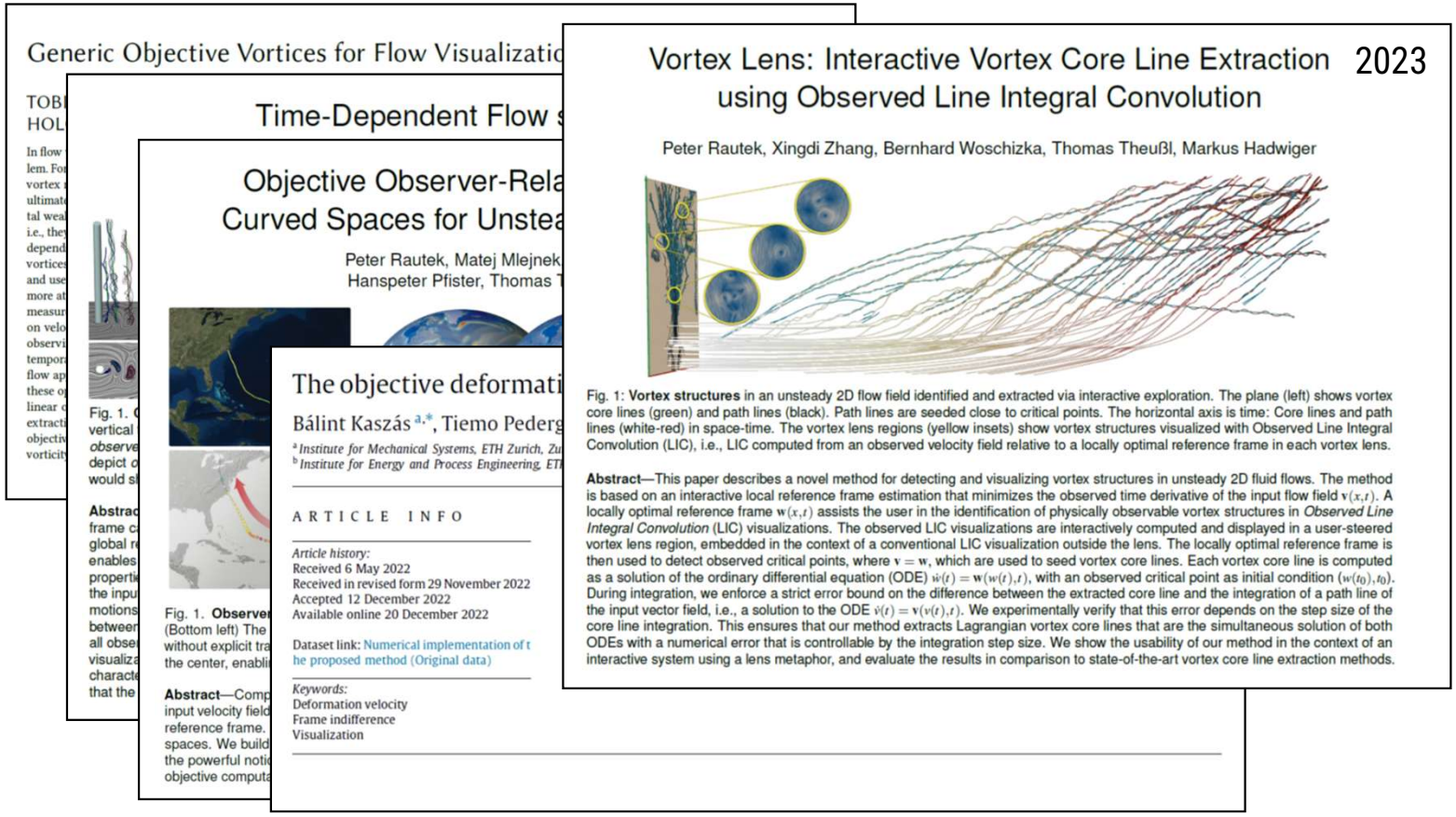
Keywords:
Deformation velocity
Frame indifference
Visualization

ABSTRACT

For an arbitrary velocity field $\mathbf{v}$ defined on a finite, fixed spatial domain, we find the closest rigid-body velocity field $\mathbf{v}_{RB}$ to $\mathbf{v}$ in the L^2 norm. The resulting deformation velocity component, $\mathbf{v}_d = \mathbf{v} - \mathbf{v}_{RB}$, turns out to be frame-indifferent and physically observable. Specifically, if $\mathbf{Q}_{RB}(t)$ is the rotation tensor describing the motion of the closest rigid body frame, then $\mathbf{v}$ is seen as $\mathbf{Q}_{RB}^T\mathbf{v}_d$ by an observer in that frame. As a consequence, the momentum, energy, vorticity, enstrophy, and helicity of the flow all become frame-indifferent when computed from the deformation velocity component $\mathbf{v}_d$.

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Observer Optimization Methods



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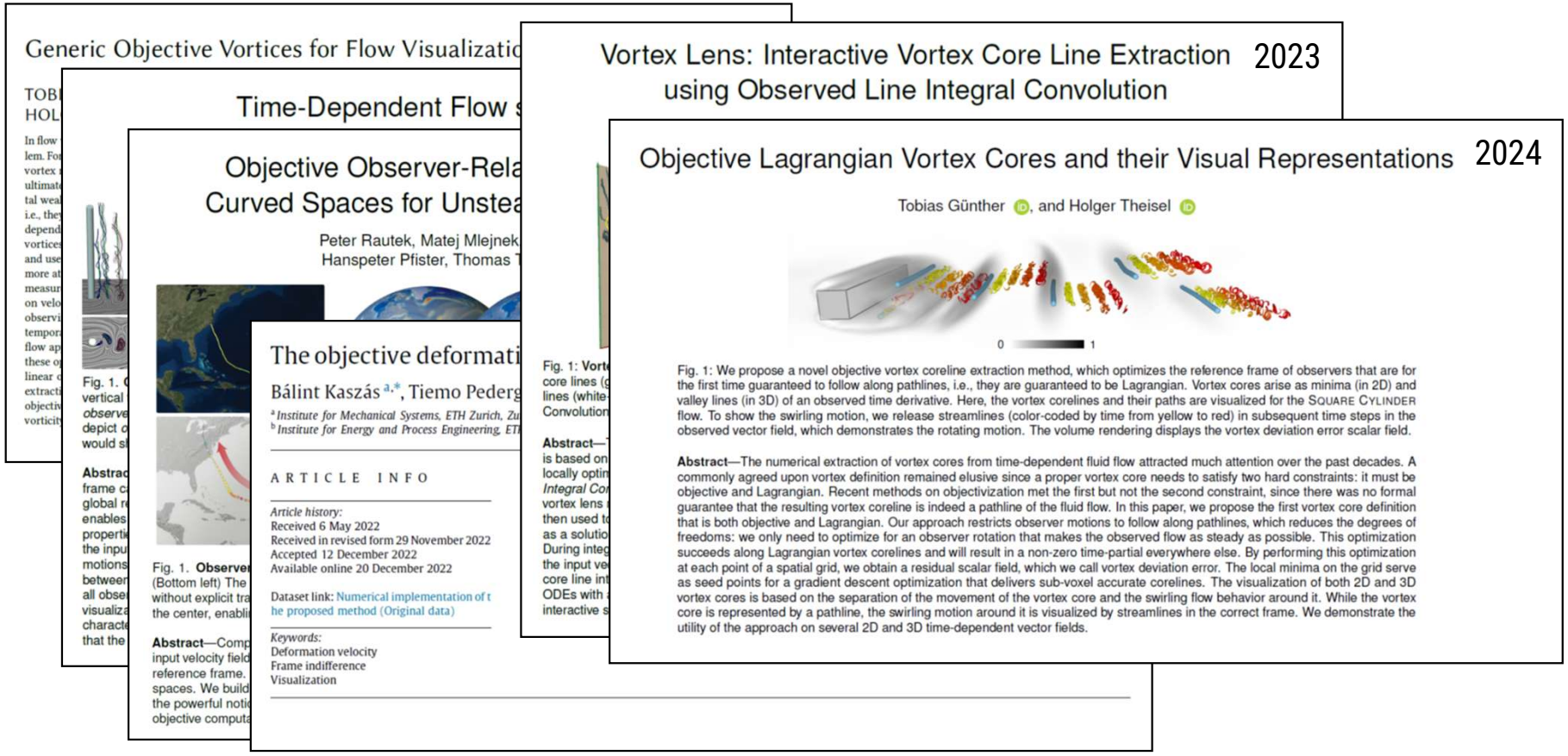
Vortex Lens: Interactive Vortex Core Line Extraction using Observed Line Integral Convolution 2023

Peter Rautek, Xingdi Zhang, Bernhard Woschizka, Thomas Theußl, Markus Hadwiger

Fig. 1: **Vortex structures** in an unsteady 2D flow field identified and extracted via interactive exploration. The plane (left) shows vortex core lines (green) and path lines (black). Path lines are seeded close to critical points. The horizontal axis is time: Core lines and path lines (white-red) in space-time. The vortex lens regions (yellow insets) show vortex structures visualized with Observed Line Integral Convolution (LIC), i.e., LIC computed from an observed velocity field relative to a locally optimal reference frame in each vortex lens.

Abstract—This paper describes a novel method for detecting and visualizing vortex structures in unsteady 2D fluid flows. The method is based on an interactive local reference frame estimation that minimizes the observed time derivative of the input flow field $\mathbf{v}(x,t)$. A locally optimal reference frame $\mathbf{w}(x,t)$ assists the user in the identification of physically observable vortex structures in *Observed Line Integral Convolution* (LIC) visualizations. The observed LIC visualizations are interactively computed and displayed in a user-steered vortex lens region, embedded in the context of a conventional LIC visualization outside the lens. The locally optimal reference frame is then used to detect observed critical points, where $\mathbf{v} = \mathbf{w}$, which are used to seed vortex core lines. Each vortex core line is computed as a solution of the ordinary differential equation (ODE) $\dot{w}(t) = \mathbf{w}(w(t),t)$, with an observed critical point as initial condition $(w(t_0),t_0)$. During integration, we enforce a strict error bound on the difference between the extracted core line and the integration of a path line of the input vector field, i.e., a solution to the ODE $\dot{v}(t) = \mathbf{v}(v(t),t)$. We experimentally verify that this error depends on the step size of the core line integration. This ensures that our method extracts Lagrangian vortex core lines that are the simultaneous solution of both ODEs with a numerical error that is controllable by the integration step size. We show the usability of our method in the context of an interactive system using a lens metaphor, and evaluate the results in comparison to state-of-the-art vortex core line extraction methods.

Observer Optimization Methods



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Vortex Lens: Interactive Vortex Core Line Extraction 2023
using Observed Line Integral Convolution

Objective Lagrangian Vortex Cores and their Visual Representations 2024

Tobias Günther, and Holger Theisel

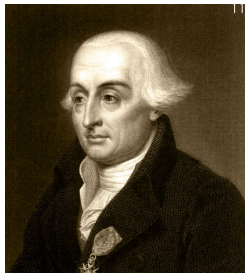
0 1

Fig. 1: We propose a novel objective vortex coreline extraction method, which optimizes the reference frame of observers that are for the first time guaranteed to follow along pathlines, i.e., they are guaranteed to be Lagrangian. Vortex cores arise as minima (in 2D) and valley lines (in 3D) of an observed time derivative. Here, the vortex corelines and their paths are visualized for the SQUARE CYLINDER flow. To show the swirling motion, we release streamlines (color-coded by time from yellow to red) in subsequent time steps in the observed vector field, which demonstrates the rotating motion. The volume rendering displays the vortex deviation error scalar field.

Abstract—The numerical extraction of vortex cores from time-dependent fluid flow attracted much attention over the past decades. A commonly agreed upon vortex definition remained elusive since a proper vortex core needs to satisfy two hard constraints: it must be objective and Lagrangian. Recent methods on objectivization met the first but not the second constraint, since there was no formal guarantee that the resulting vortex coreline is indeed a pathline of the fluid flow. In this paper, we propose the first vortex core definition that is both objective and Lagrangian. Our approach restricts observer motions to follow along pathlines, which reduces the degrees of freedoms: we only need to optimize for an observer rotation that makes the observed flow as steady as possible. This optimization succeeds along Lagrangian vortex corelines and will result in a non-zero time-partial everywhere else. By performing this optimization at each point of a spatial grid, we obtain a residual scalar field, which we call vortex deviation error. The local minima on the grid serve as seed points for a gradient descent optimization that delivers sub-voxel accurate corelines. The visualization of both 2D and 3D vortex cores is based on the separation of the movement of the vortex core and the swirling flow behavior around it. While the vortex core is represented by a pathline, the swirling motion around it is visualized by streamlines in the correct frame. We demonstrate the utility of the approach on several 2D and 3D time-dependent vector fields.

Phase Spaces of Observers

Lagrangian / Hamiltonian Mechanics (1)



Joseph-Louis Lagrange (1736-1813)

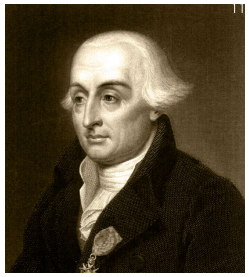
Configuration space (manifold)

- Encodes all physical *degrees of freedom* as independent variables
- k-dimensional manifold for k degrees of freedom

Phase space (manifold)

- Configuration space plus derivatives needed to define energy (cost) function
- 2k-dimensional for k degrees of freedom; (2k+1)-dimensional with time dimension

Lagrangian / Hamiltonian Mechanics (2)



Joseph-Louis Lagrange (1736-1813)



William Rowan Hamilton (1805-1865)

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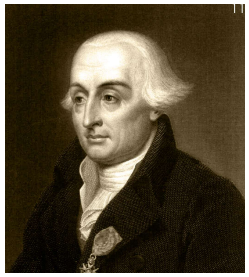
Lagrangian phase space

- Generalized coords & *generalized velocities*
- Tangent bundle of the configuration space

Hamiltonian phase space

- Generalized coords & *generalized momenta*
- Cotangent bundle of the configuration space

Lagrangian / Hamiltonian Mechanics (2)



Joseph-Louis Lagrange (1736-1813)



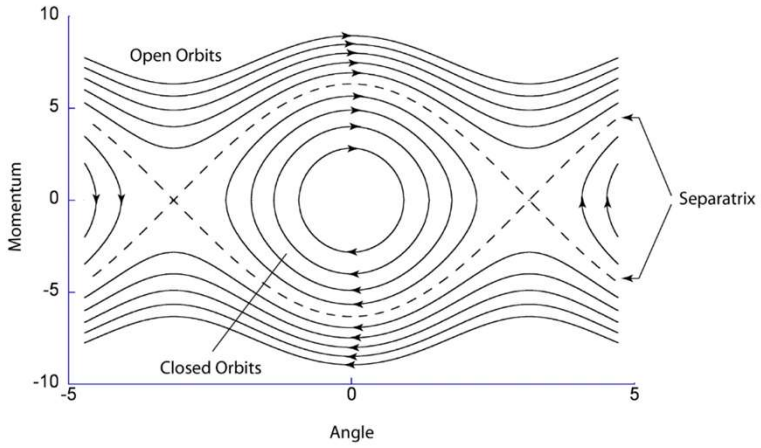
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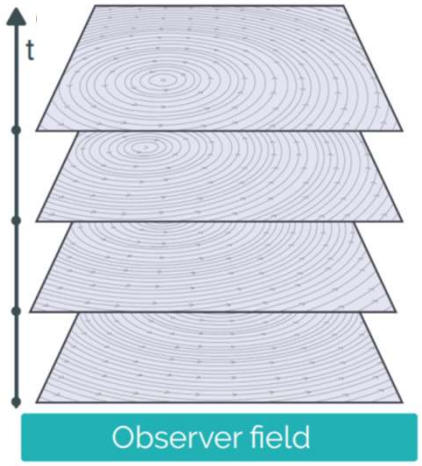


Hamiltonian phase space

- Generalized coords & *generalized momenta*
- Cotangent bundle of the configuration space

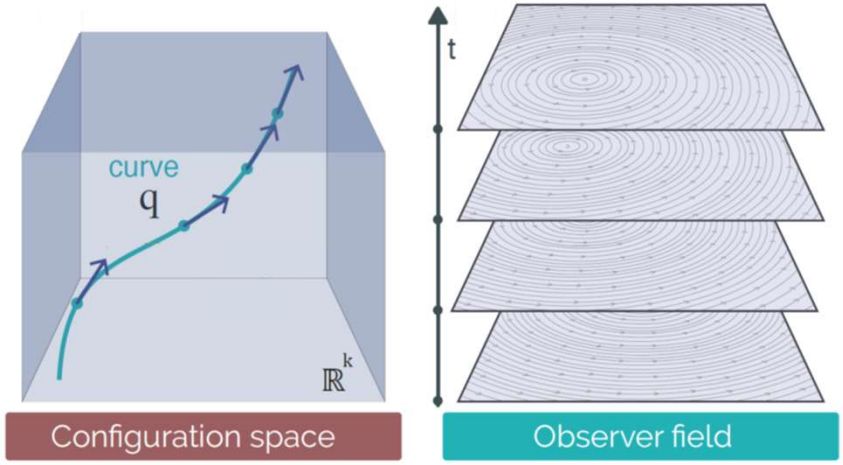
Observer Configuration Space

k-dimensional configuration space: *k* physical *degrees of freedom* (independent variables)



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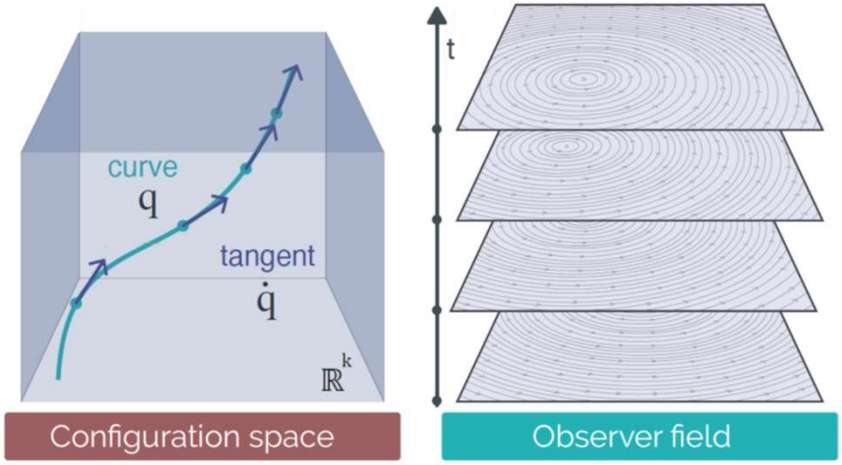


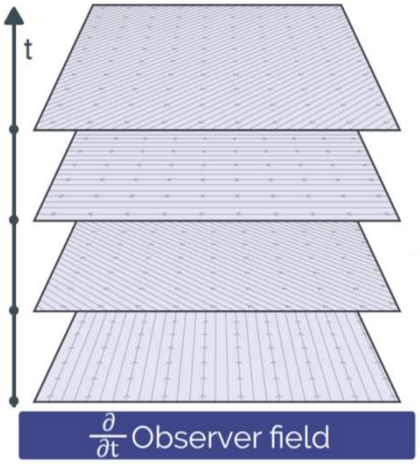
$$\mathbf{u}(x,t) = \sum_{i=1}^{k} q^i(t)\,\mathbf{e}_i(x,t)$$

$$\mathbf{q}(t) = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}(t)$$

Observer Configuration Space

k-dimensional configuration space: *k* physical *degrees of freedom* (independent variables)





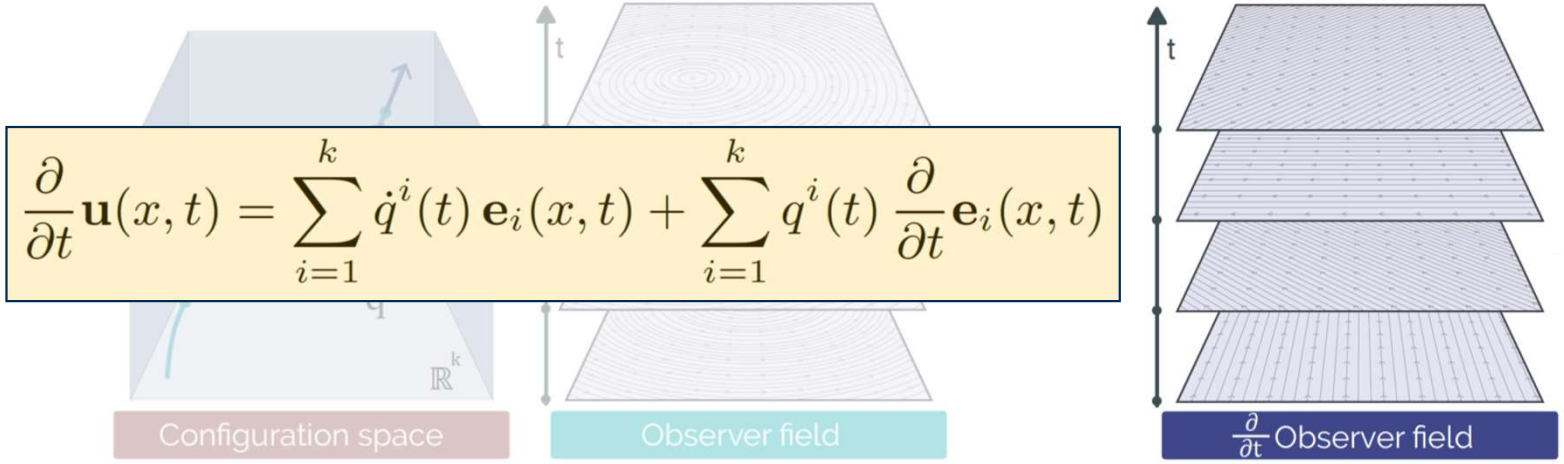
$$\mathbf{q}(t) = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}(t)$$

$$\dot{\mathbf{q}}(t) = \begin{pmatrix} \dot{q}^1 \\ \dot{q}^2 \\ \dot{q}^3 \end{pmatrix}(t) = \frac{d}{dt}\begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}(t)$$

Observer Configuration Space

k-dimensional configuration space: *k* physical *degrees of freedom* (independent variables)

$$\frac{\partial}{\partial t}\mathbf{u}(x,t) = \sum_{i=1}^{k} \dot{q}^i(t)\,\mathbf{e}_i(x,t) + \sum_{i=1}^{k} q^i(t)\,\frac{\partial}{\partial t}\mathbf{e}_i(x,t)$$



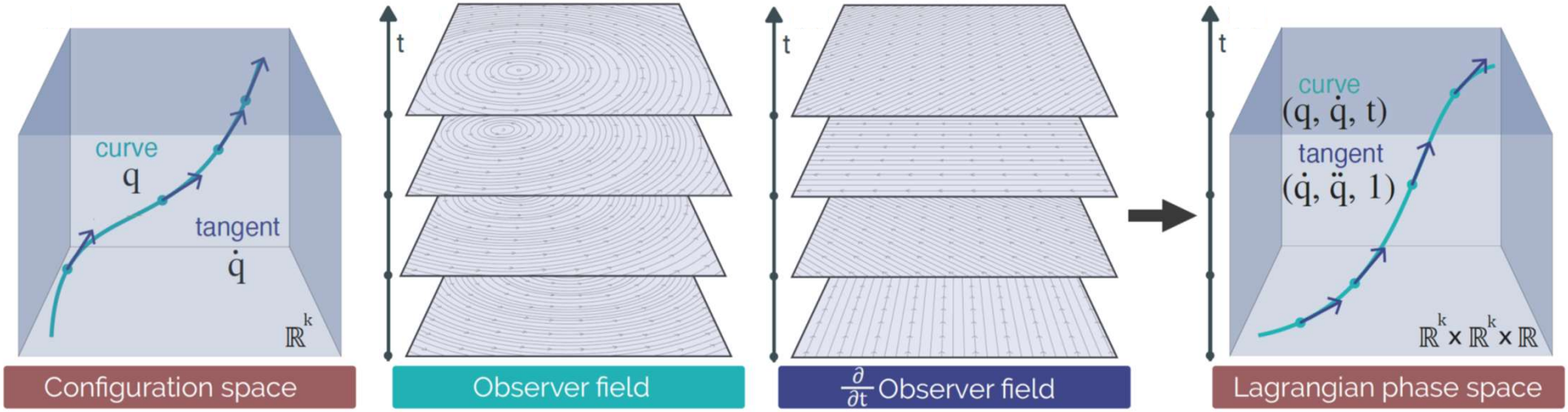
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Variational Computation of Observers

Lagrangian Observer Phase Space

Phase space becomes domain of scalar cost function: the Lagrangian



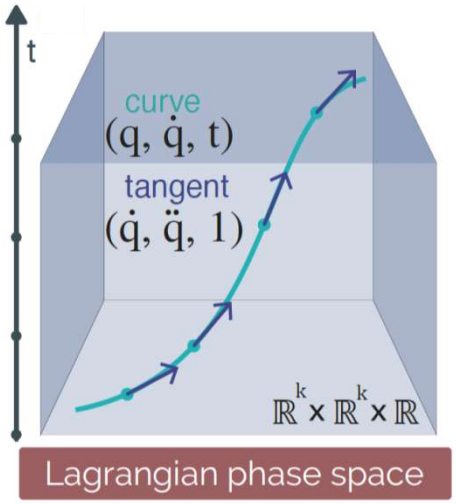
The Lagrangian

Phase space becomes domain of scalar cost function: the Lagrangian

$$L\colon \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R},$$
$$(\mathbf{q}, \dot{\mathbf{q}}, t) \mapsto L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

Find curve in phase space that minimizes the *action* (integral)

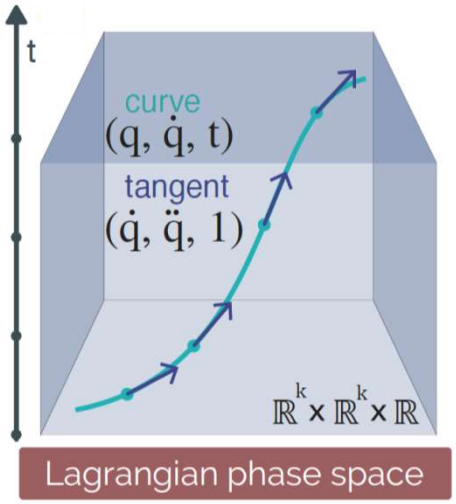
$$S[\mathbf{q}] = \int_{t_a}^{t_b} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, dt$$



The Lagrangian

Phase space becomes domain of scalar cost function: the Lagrangian

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Find curve in phase space that minimizes the *action* (integral)

$$S[\mathbf{q}] = \int_{t_a}^{t_b} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, dt$$

Remember observed time derivative $\frac{D}{Dt}(\boldsymbol{v} - \boldsymbol{w}) = \left(\frac{\partial}{\partial t} + \mathcal{L}_{\boldsymbol{w}}\right)(\boldsymbol{v} - \boldsymbol{w})$

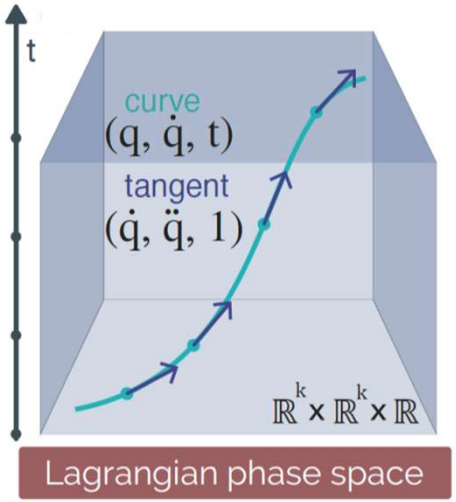
Calculus of Variations

Euler-Lagrange operator

k-vector function (gradient) on curve

$$\psi = \frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}$$

(vector-valued) second-order ODE

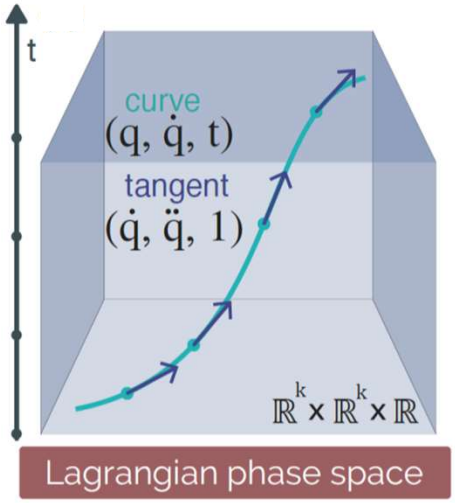


Euler-Lagrange equation:

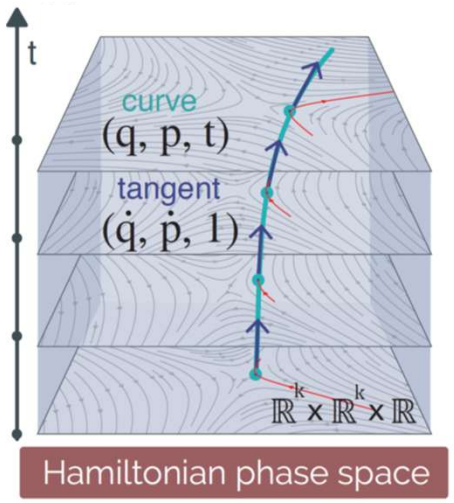
operator vanishes for optimum

$$\psi = 0$$

Lagrangian vs. Hamiltonian



Legendre transform



$$t \mapsto (\mathbf{q}, \dot{\mathbf{q}}, t)(t)$$

$$L\colon \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$$

$$t \mapsto (\mathbf{q}, \mathbf{p}, t)(t)$$

$$H\colon \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$$

Hamiltonian Vector Fields

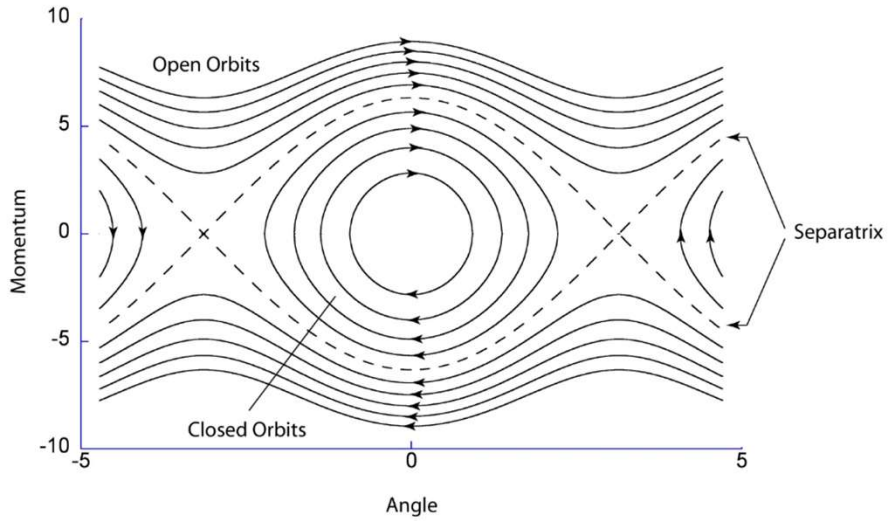
Hamilton's equations

$$\dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q},\mathbf{p},t),$$

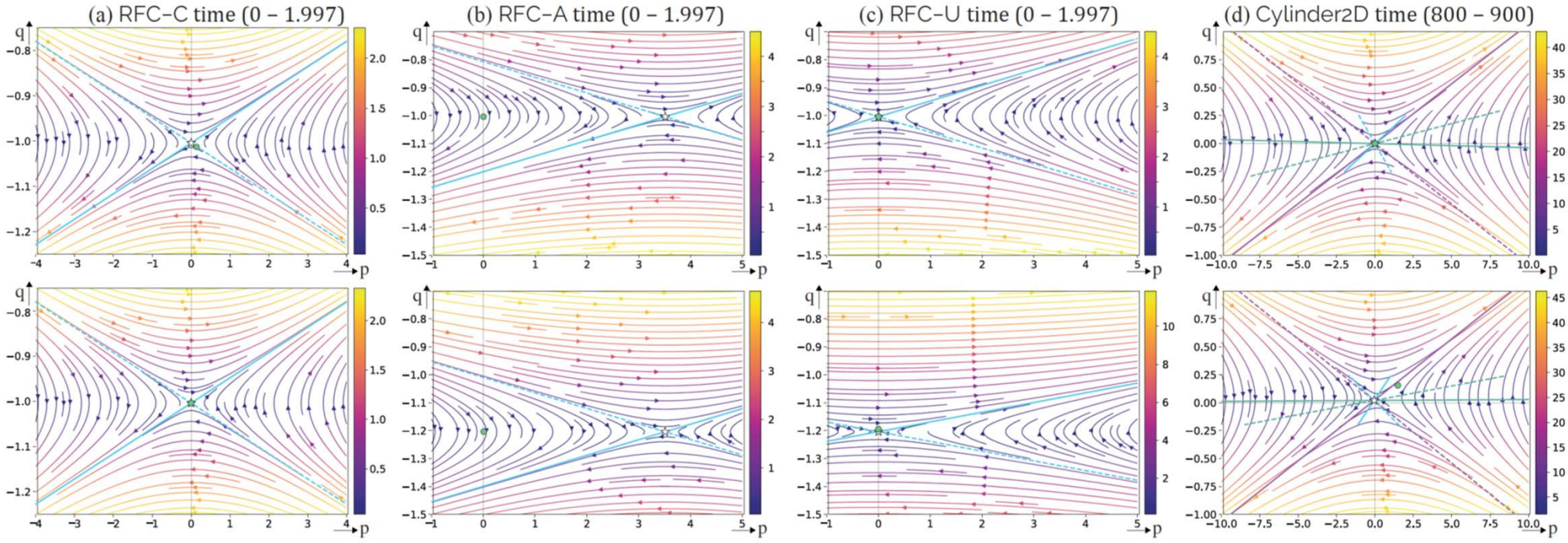
$$\dot{\mathbf{p}}(\mathbf{q},\mathbf{p},t) = -\frac{\partial H}{\partial \mathbf{q}}(\mathbf{q},\mathbf{p},t).$$

Hamiltonian vector field

$$X_H(\mathbf{q},\mathbf{p},\tau) := \begin{pmatrix} \dot{\mathbf{q}}(\mathbf{q},\mathbf{p},\tau) \\ \dot{\mathbf{p}}(\mathbf{q},\mathbf{p},\tau) \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial \mathbf{p}} \\ -\frac{\partial H}{\partial \mathbf{q}} \end{pmatrix} (\mathbf{q},\mathbf{p},\tau).$$

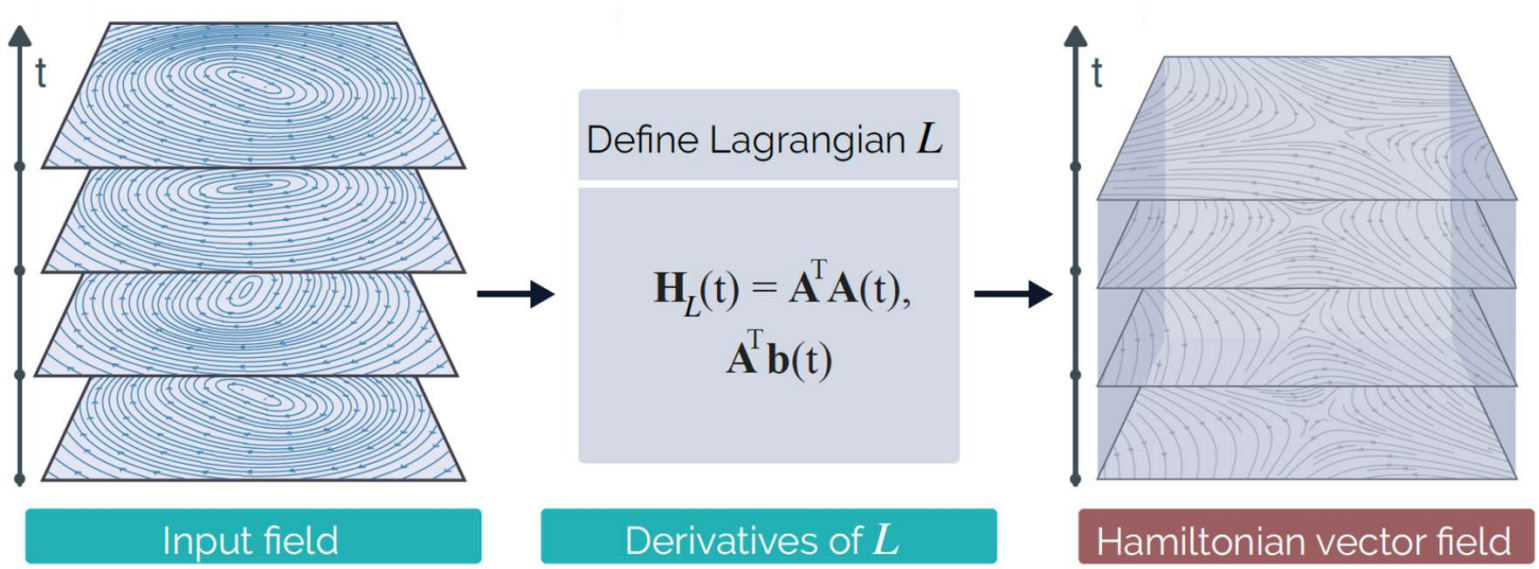


Hamiltonian Vector Fields of Observers



Our Lagrangian-Hamiltonian Framework (1)

Compute *Hamiltonian vector field* corresponding to input flow field



Our Lagrangian-Hamiltonian Framework (2)

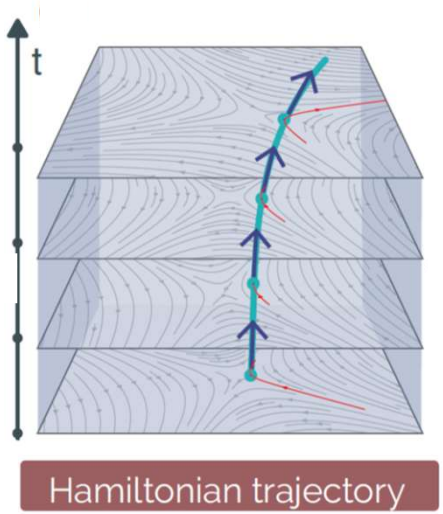
1. Define Lagrangian for specific cost
2. Directly gives Hamiltonian vector field

Determine specific optimal observer

- How initial or boundary values are chosen and handled
- With or without gradient descent for refinement

Novel example applications

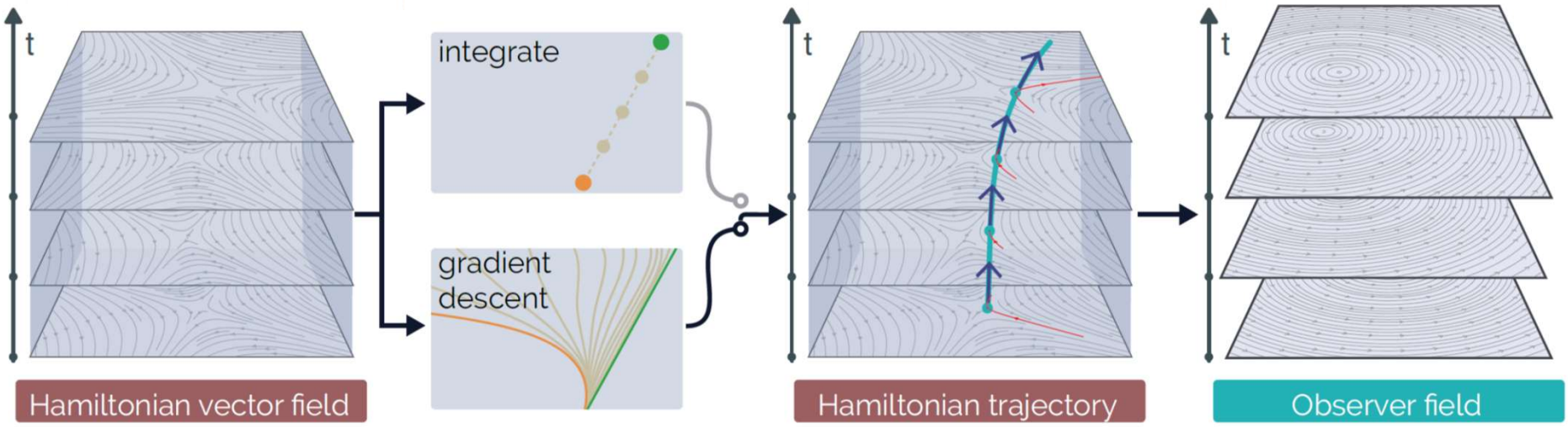
- Vortex core topology depends on time window

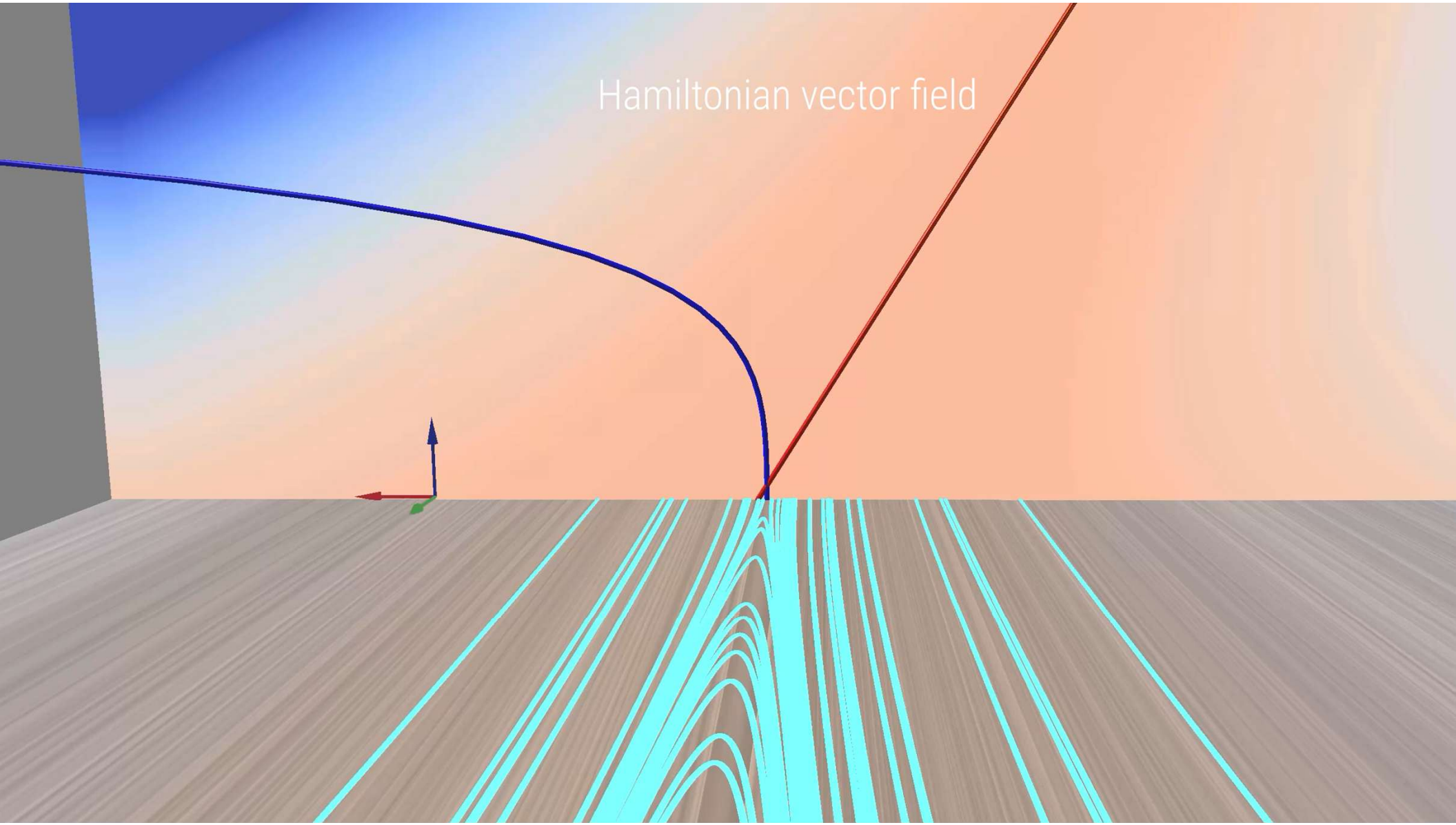


Our Lagrangian-Hamiltonian Framework (3)

Integrate trajectory (Runge-Kutta or symplectic integrator)

Functional gradient descent



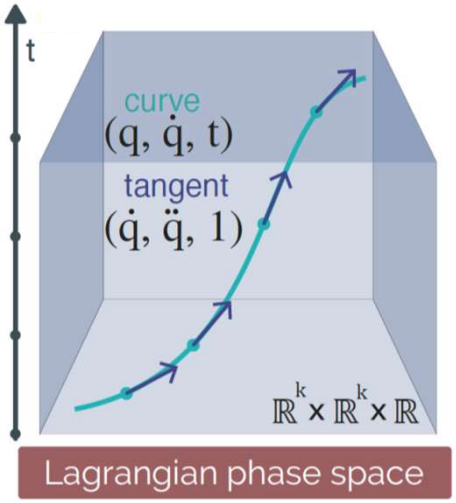

Hamiltonian vector field

Specific Optimal Observers (1)

All Hamiltonian trajectories are optimal!

Specific optimal observer chosen via

- Initial value
  (phase space coord where integration starts)
- Boundary conditions
  (free boundary/natural boundary conditions)

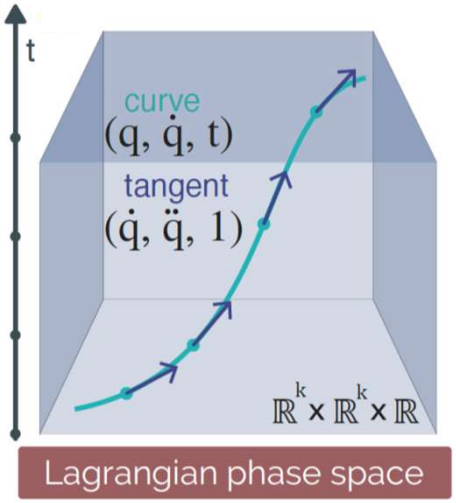


Specific Optimal Observers (2)

Free boundary:

Main choice are **start time** and **end time**

$$S[\mathbf{q}] = \int_{t_a}^{t_b} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, dt$$



- Fixed time window: start and end time given
- Progressive in time: gradient descent can adjust time window on the fly

Example:
Interactive Exploration via Vortex Lens

Minimize Functional

$$S[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)\, dt$$

Euler-Lagrange Equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

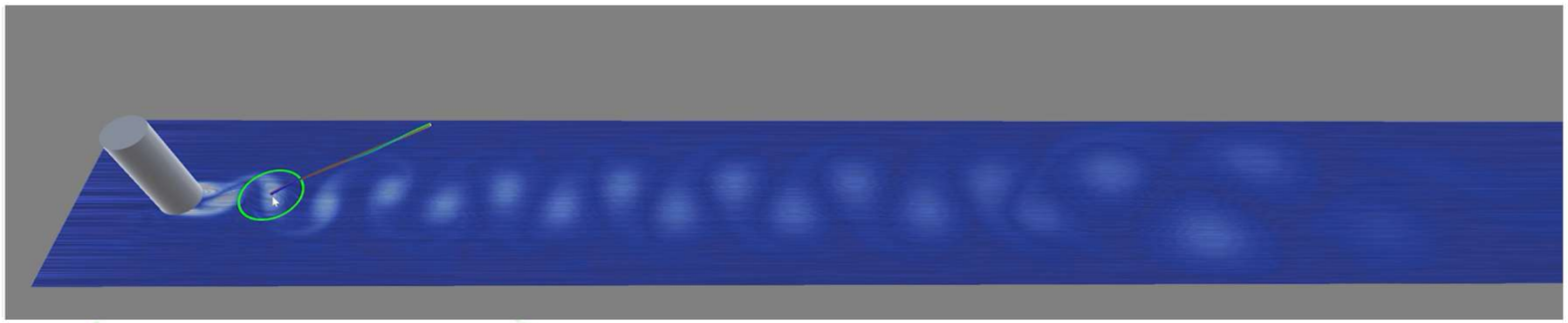
Legendre Transform

$$p := \frac{\partial L}{\partial \dot{q}}$$

Hamiltonian System

$$(q, p)$$

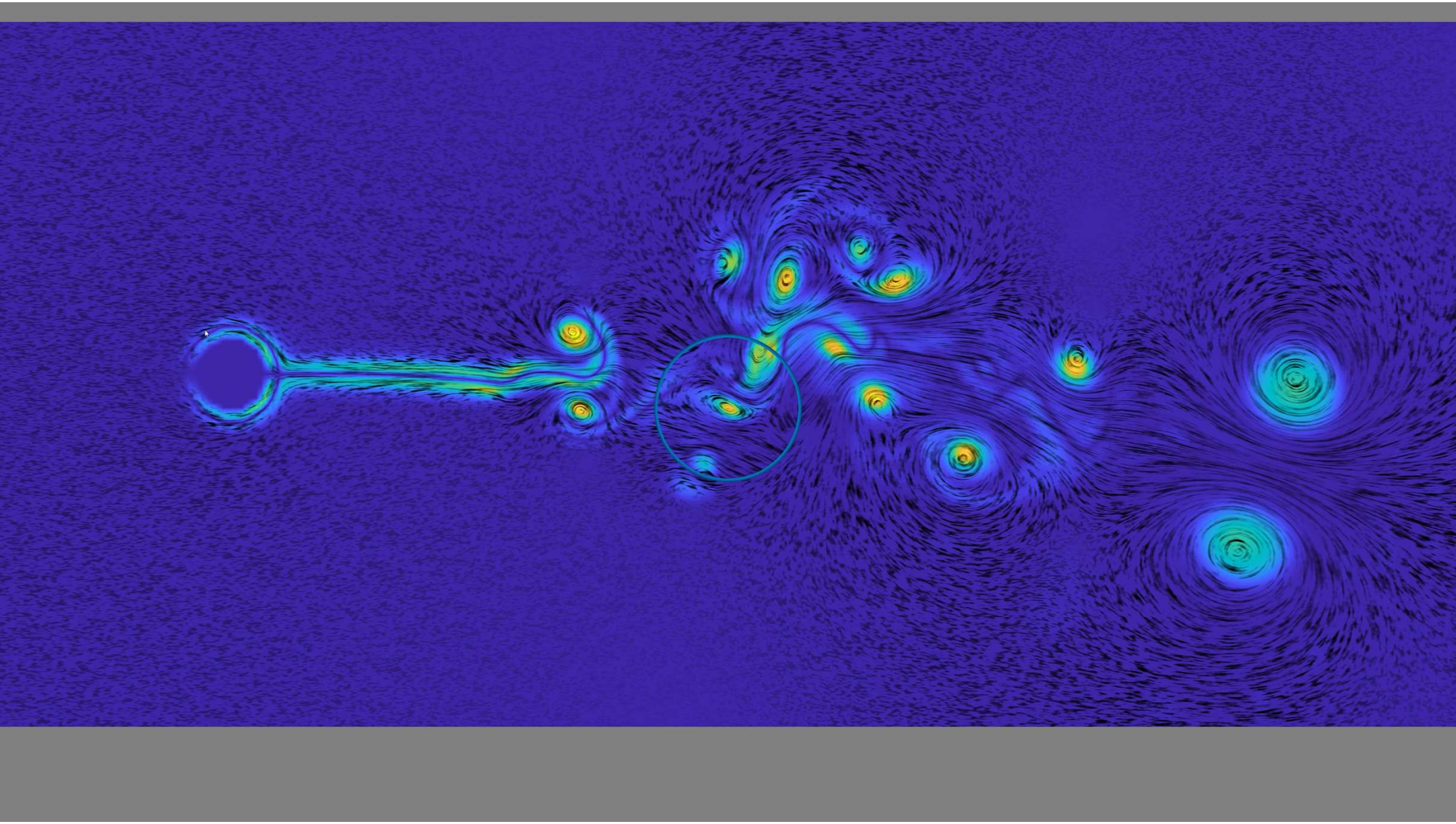
Integrate Hamiltonian Vector Field



Hamiltonian phase space

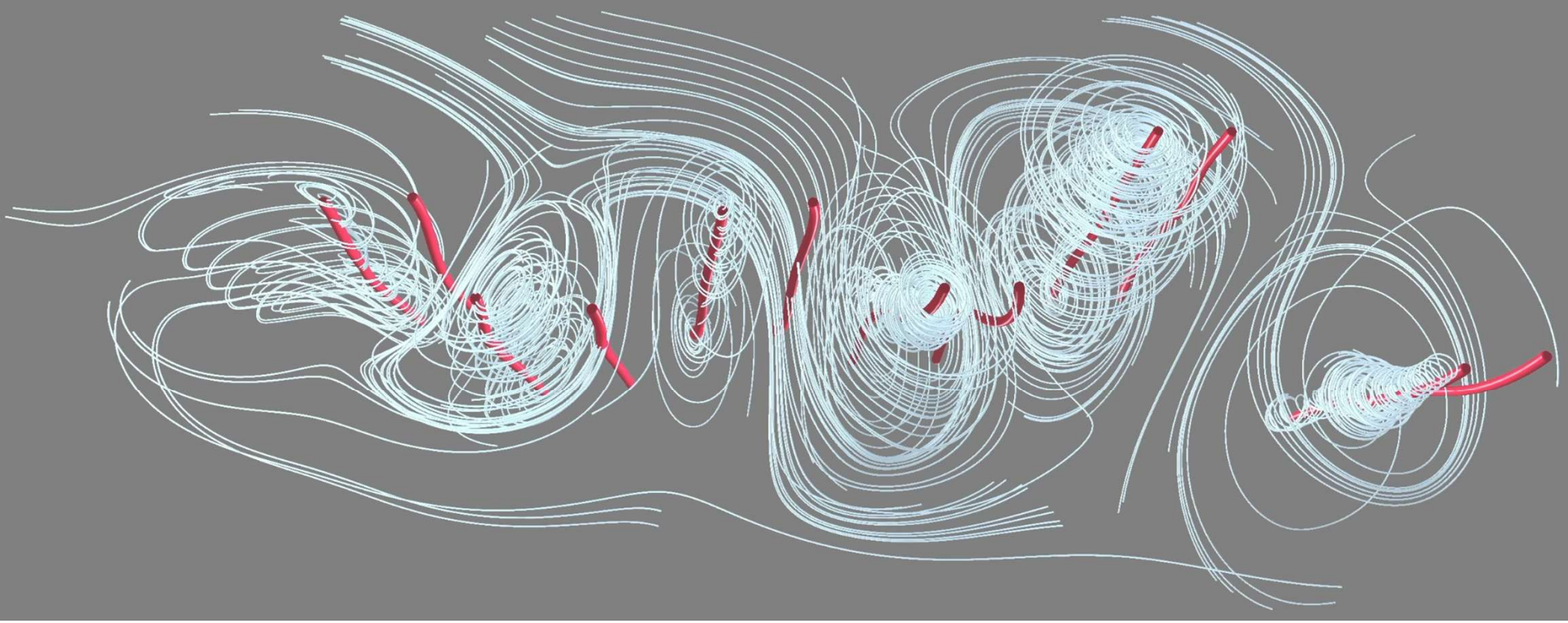
Integrate Hamiltonian Vector Field

Follow critical curve in observed field



Vortex Core Manifolds in 3D Flow

Vortex Cores are Spacetime 2-Manifolds



Variational Method

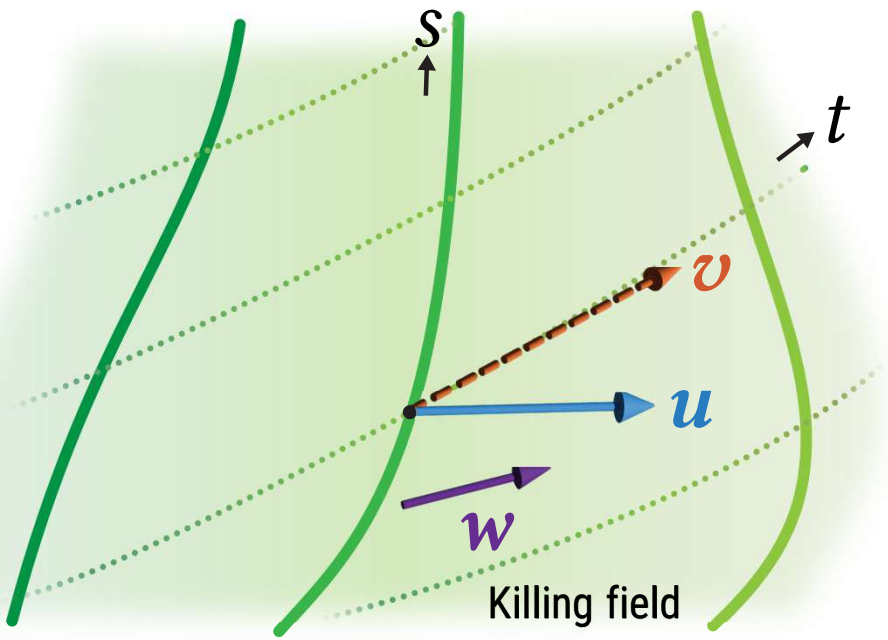
Generalization of steady flow case

- Vortex core line → spacetime 2-manifold
- Killing field $\boldsymbol{w}$ → **approximately** Killing $\boldsymbol{u}$

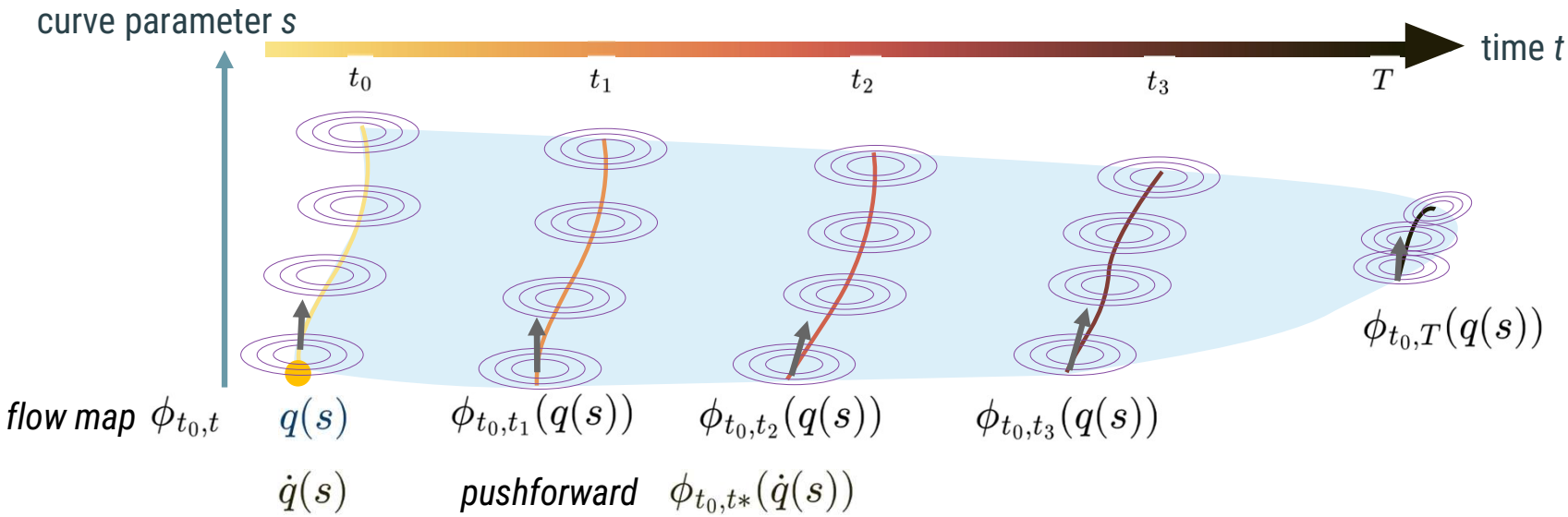
Optimal curve from Euler-Lagrange

- Flow is known:

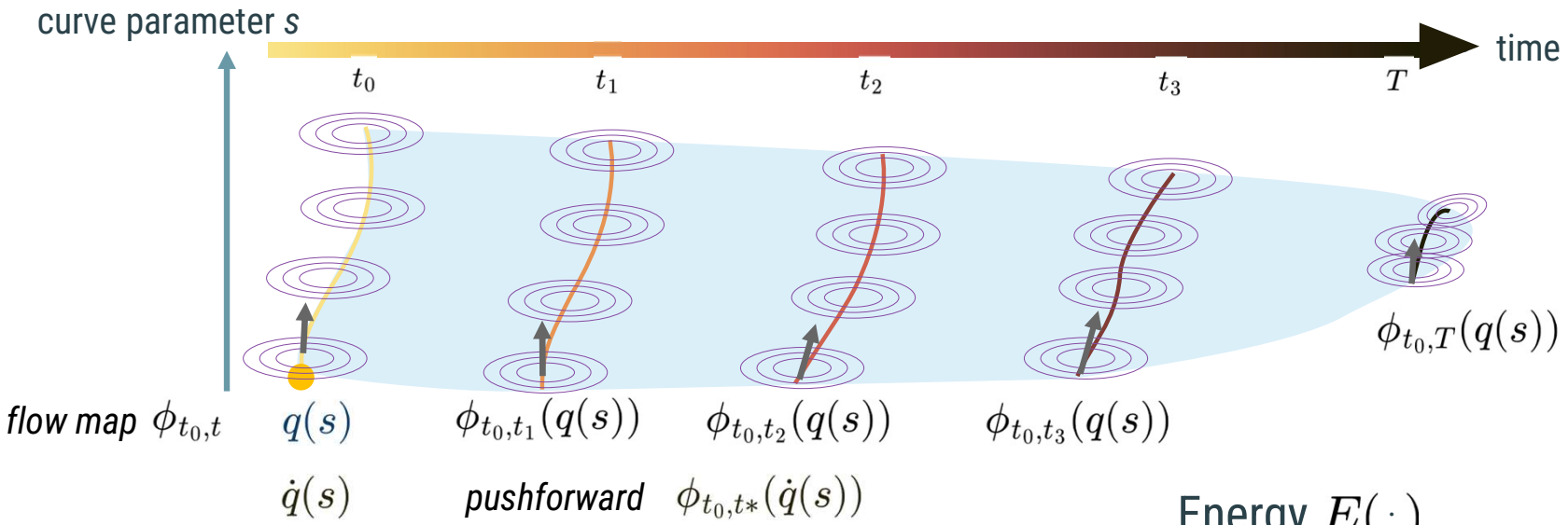
  Curve gives optimal 2-manifold



Pre-Integrated Lagrangian: Curve gives Surface



Pre-Integrated Lagrangian: Curve gives Surface



$$L(q,\dot{q},s) = \int_{t_0}^{T} E(\phi_{t_0,t}(q), \phi_{t_0,t*}(\dot{q}), t)\, dt$$

Energy $E(\cdot)$

magnitude of deformation,

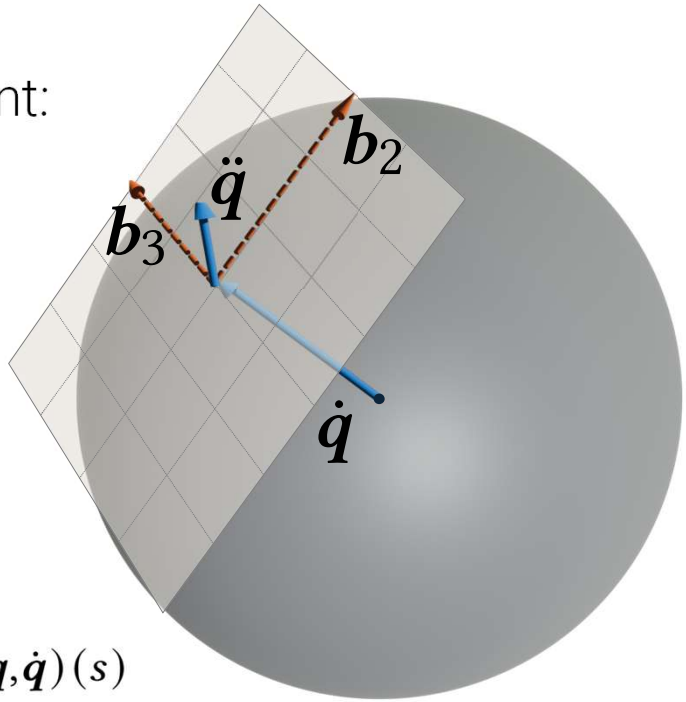
generic vortex criterion (e.g., IVD),

additional regularization

Solving the Euler-Lagrange Equation

Lagrangian independent of $\|\dot{\boldsymbol{q}}\|$ + arc length constraint:
Solve for $\ddot{\boldsymbol{q}}$ in plane orthogonal to core line tangent $\dot{\boldsymbol{q}}$

Each step is simple 2x2 linear system solve
2x2 Hessian matrix in spherical coordinates

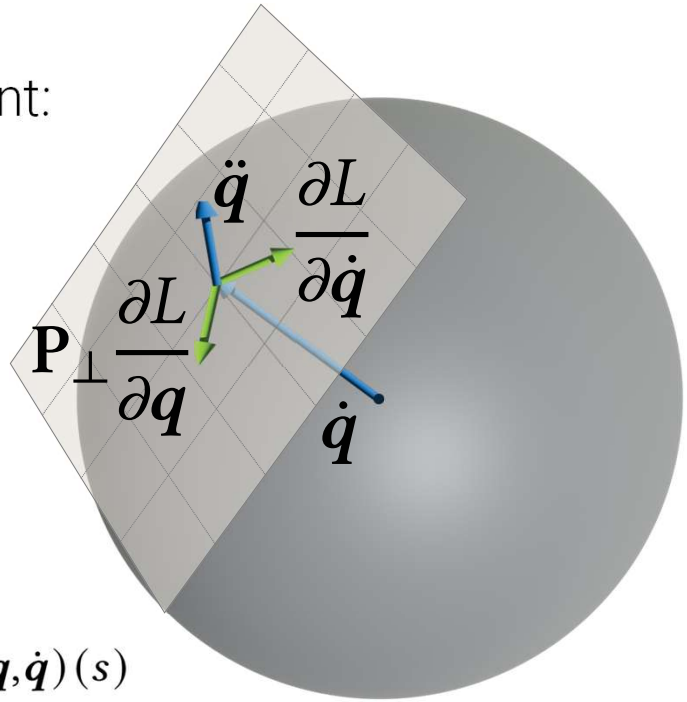


$$\ddot{\boldsymbol{q}}(s) = \left[\left(\boldsymbol{b}_2 | \boldsymbol{b}_3\right) \left(\bar{\mathbf{H}} + 2\lambda(s)\, \mathbf{I}^{2\times 2} \right)^{-1} \left(\boldsymbol{b}_2 | \boldsymbol{b}_3\right)^T \frac{\partial L}{\partial \boldsymbol{q}} \right]_{(\boldsymbol{q},\dot{\boldsymbol{q}})(s)}$$

Solving the Euler-Lagrange Equation

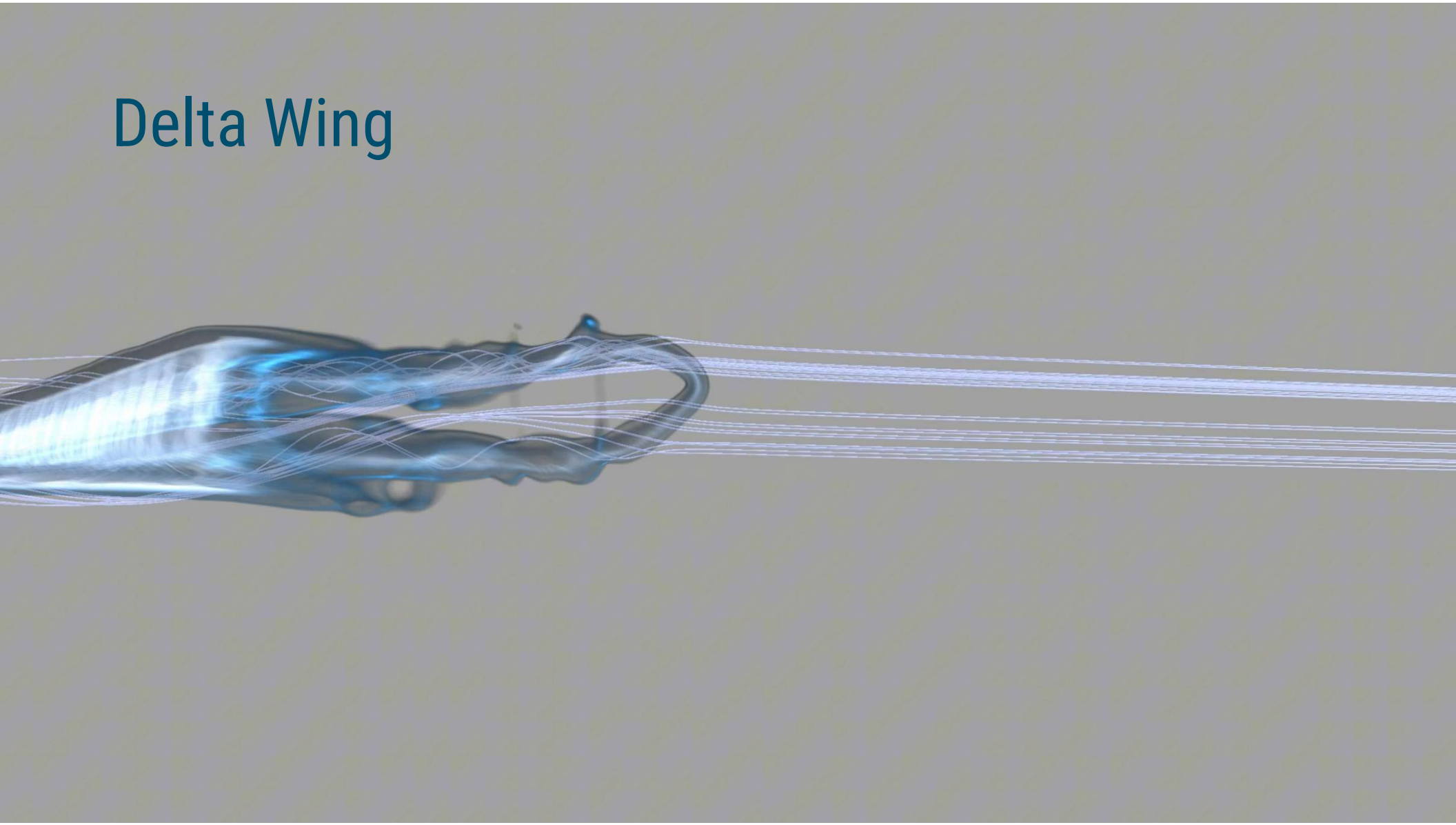
Lagrangian independent of $\|\dot{\boldsymbol{q}}\|$ + arc length constraint:
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2x2 Hessian matrix in spherical coordinates

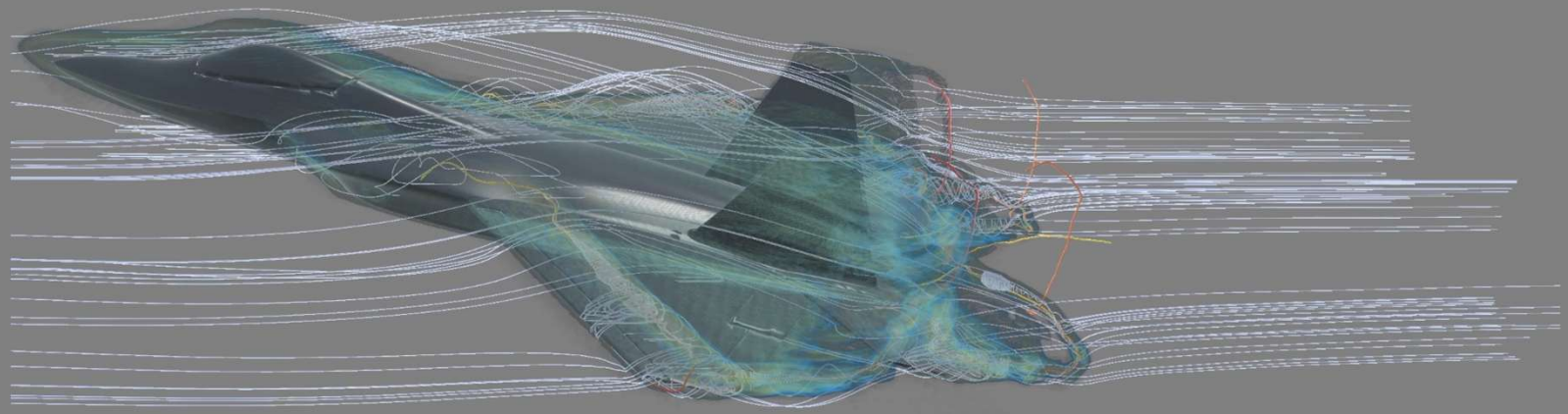


$$\ddot{\boldsymbol{q}}(s) = \left[\left(\boldsymbol{b}_2|\boldsymbol{b}_3\right) \left(\bar{\mathbf{H}} + 2\lambda(s)\,\mathbf{I}^{2\times 2}\right)^{-1} \left(\boldsymbol{b}_2|\boldsymbol{b}_3\right)^{T} \frac{\partial L}{\partial \boldsymbol{q}} \right]_{(\boldsymbol{q},\dot{\boldsymbol{q}})(s)}$$

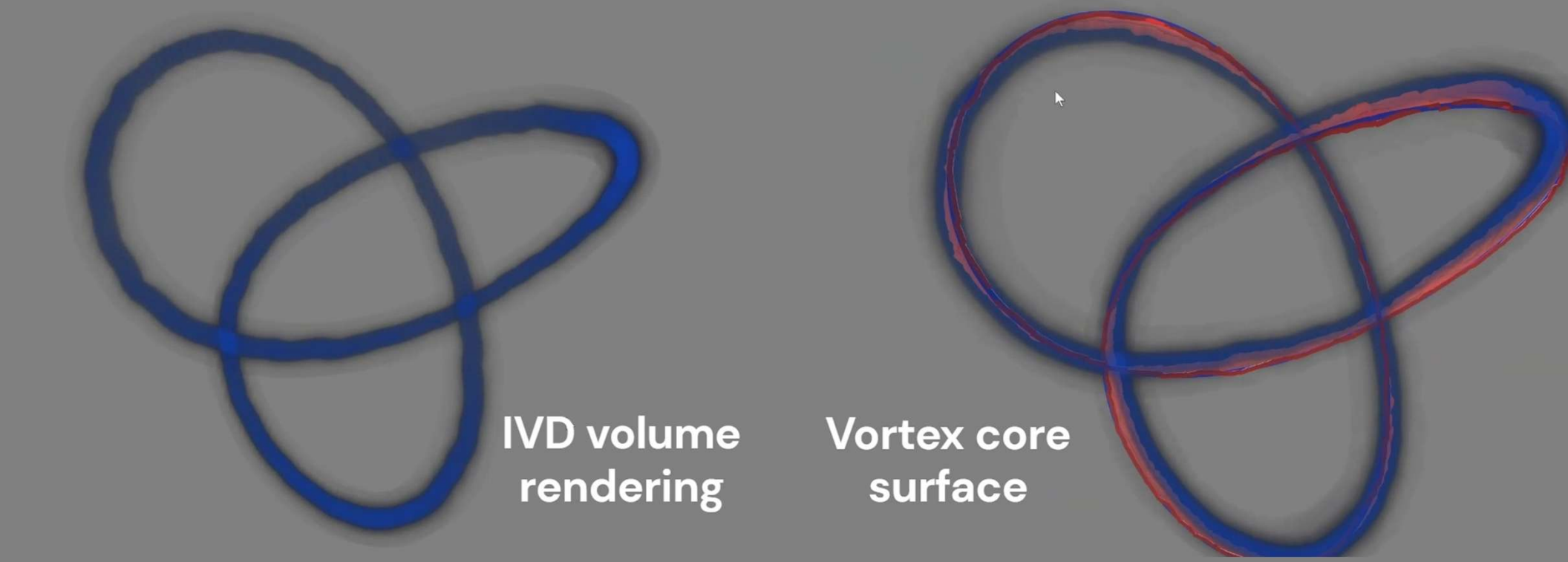
Delta Wing



Raptor



Trefoil Knot



Conclusions

Can unify and extend all existing observer optimization methods

- 2D and 3D flows, Killing fields and approximate Killing fields
- Different choices for choosing optimal observer
- Hamiltonian gives insight (e.g., intrinsically steady flow fields)

Variational vortex cores in 3D flow

- Curve optimization gives optimal spacetime surface
- Enabled by flow map and its pushforward

Thank You!

Special thanks for contributions to

Xingdi Zhang, Amani Ageeli, Thomas Theußl, Peter Rautek

vccvisualization.org

