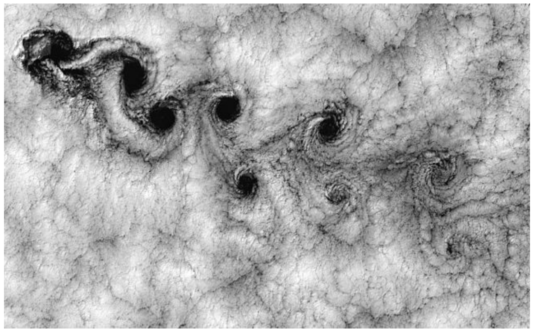
Generic Variational Spacetime Optimization of Vortex Core Manifolds

Xingdi Zhang, Peter Rautek, Markus Hadwiger
KAUST

Flow Visualization & Vortex Extraction



geophysical meteorology[1]

airfoil design[3]

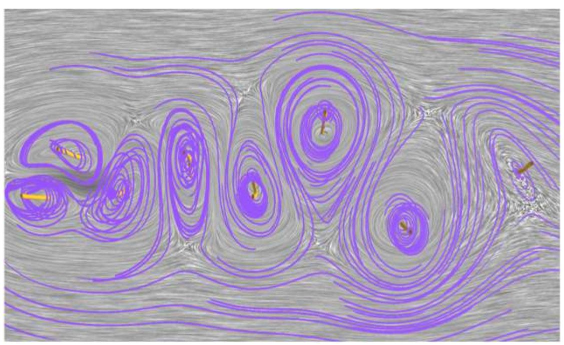
oceanography and eddy detection[2]

1. [image by Landsat, from wikimedia]
2. [image by Olrat, from shutterstock]
3. [image from Wikipedia]

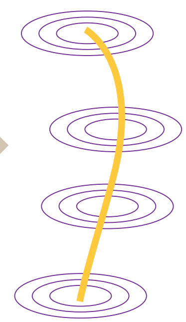
Flow Visualization & Vortex Extraction

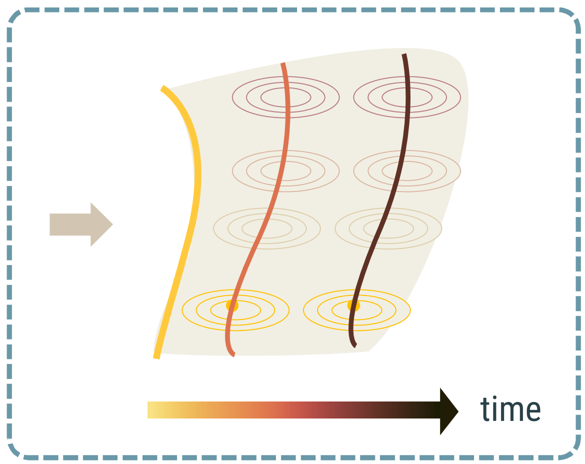
No universal agreement on what constitutes a vortex

- In a reference frame following a vortex, streamlines should be circles/spirals
- 3D unsteady flow: Reference frame cannot perfectly follow a vortex



vortex street





Our focus:

General 3D unsteady flow

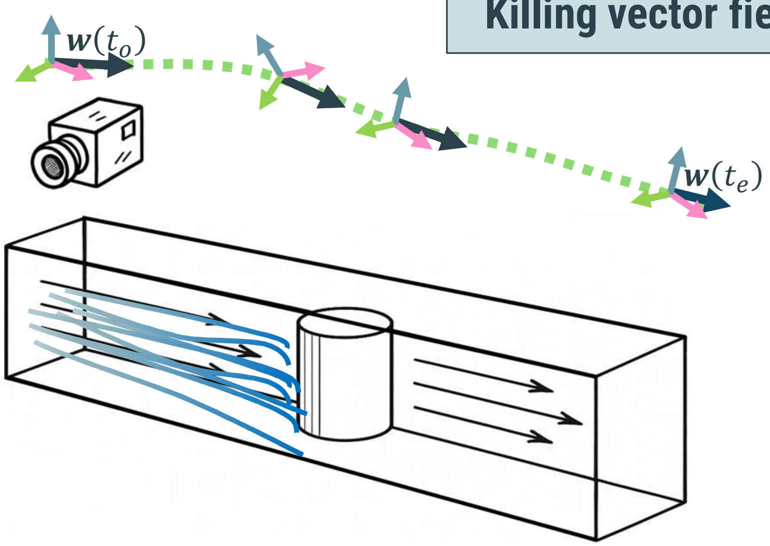
Reference Frames / Observers and Killing Fields

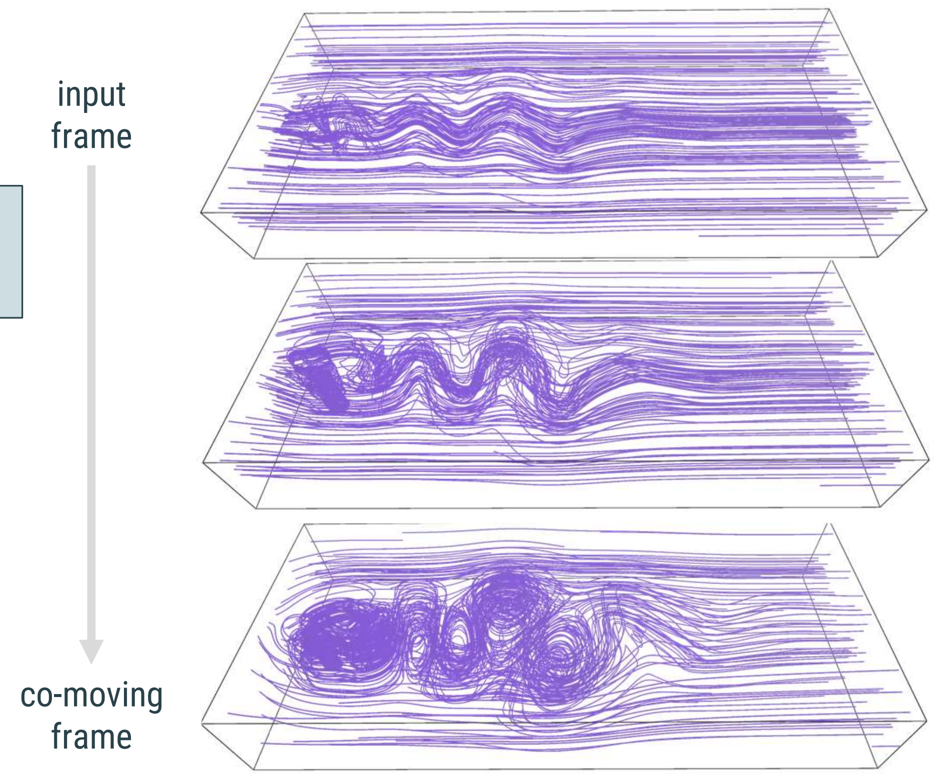
Observer motion is rigid transformation

$$\boldsymbol{x}^* = \boldsymbol{Q}(t)\,\boldsymbol{x} + \boldsymbol{c}(t)$$

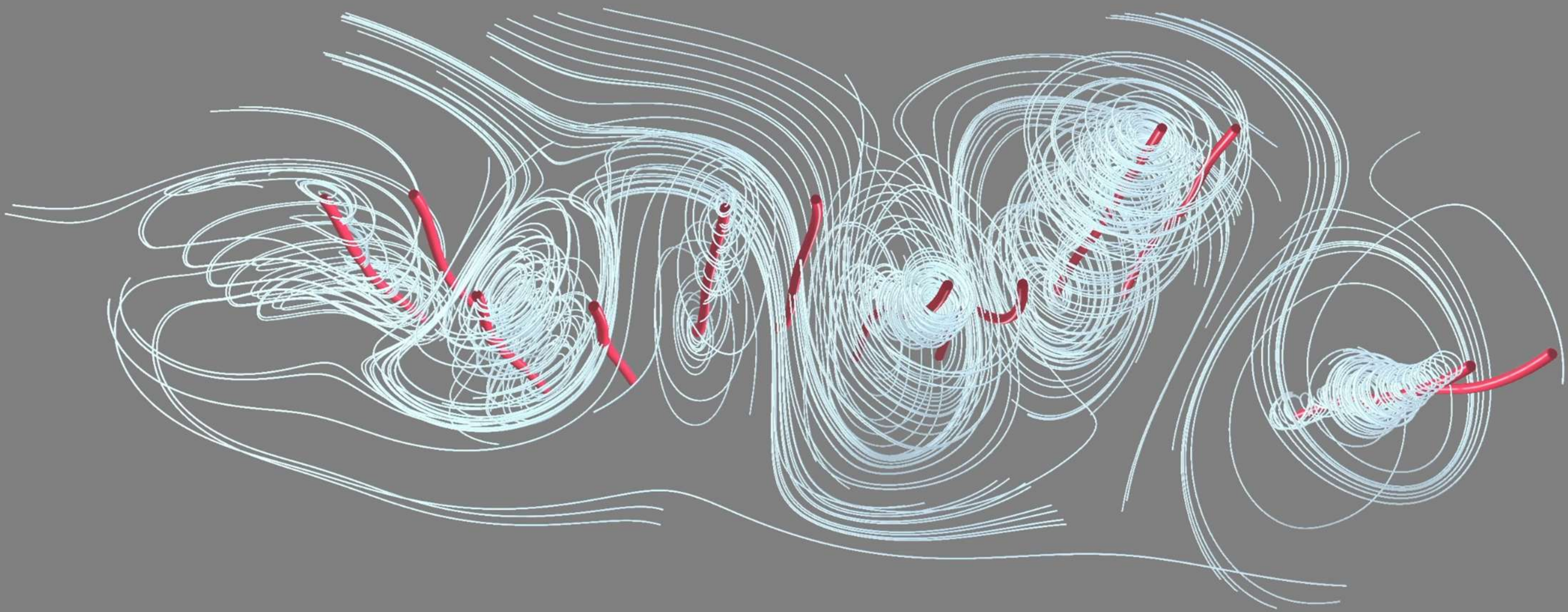
Derivative:

Observer velocity field $\boldsymbol{w}(t)$
Killing vector field





Vortex Cores are Spacetime 2-Manifolds



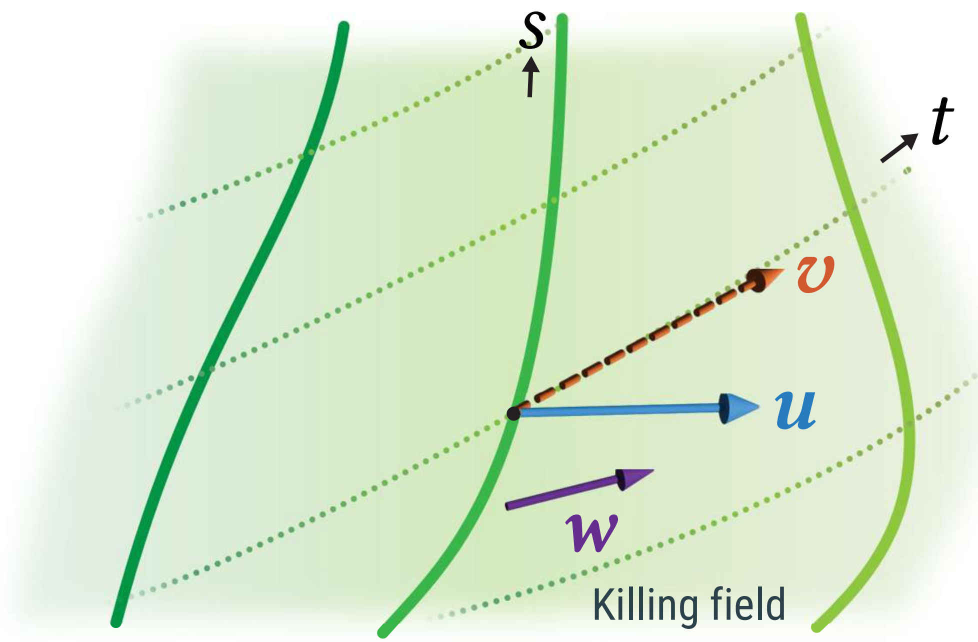
Our Method

Generalization of steady flow case

- Vortex core line → spacetime 2-manifold
- Killing field $\boldsymbol{w}$ → **approximate** Killing field $\boldsymbol{u}$

Euler-Lagrange solve for optimal **curve**

- Flow is known: **Curve gives optimal 2-manifold**



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Key Idea #1

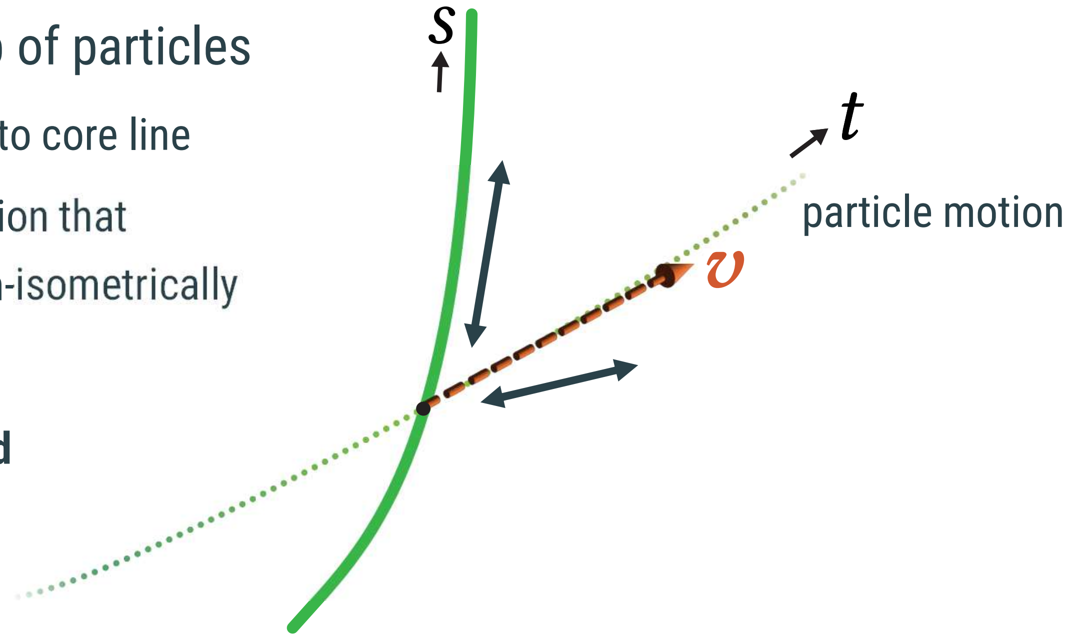
Search for Minimally Deforming Vortex Core Line

Spacetime Search: Minimally Deforming Core Line

Vortex core line made up of particles

- Arbitrary motion *tangential* to core line
- Minimal *non-tangential* motion that transports the core line non-isometrically

Non-tangential motion should be "*as-Killing-as-possible*"



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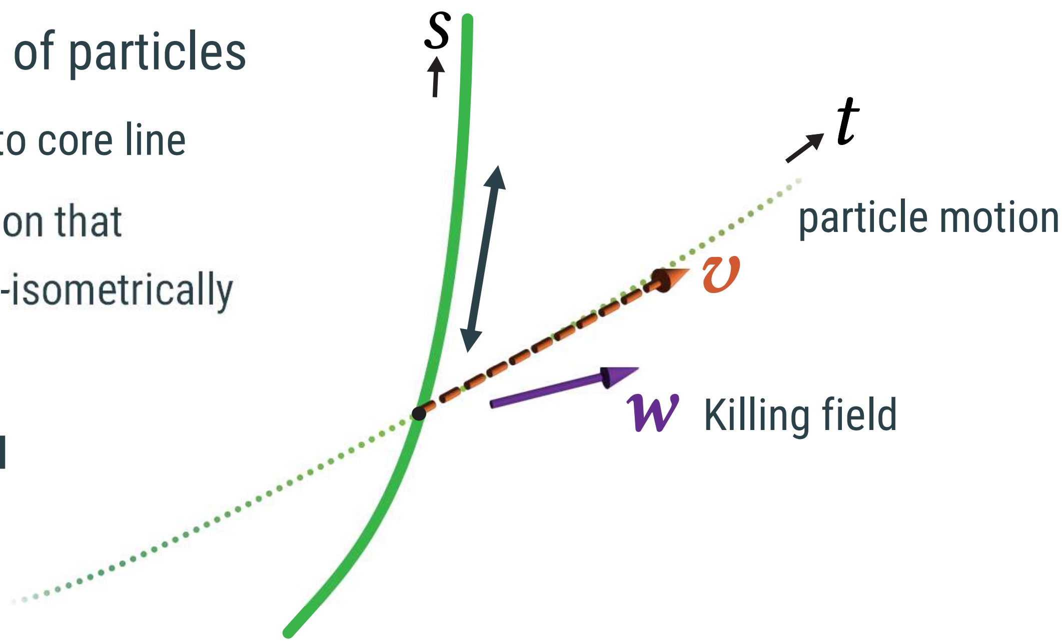


Spacetime Search: Minimally Deforming Core Line

Core line sweeps out

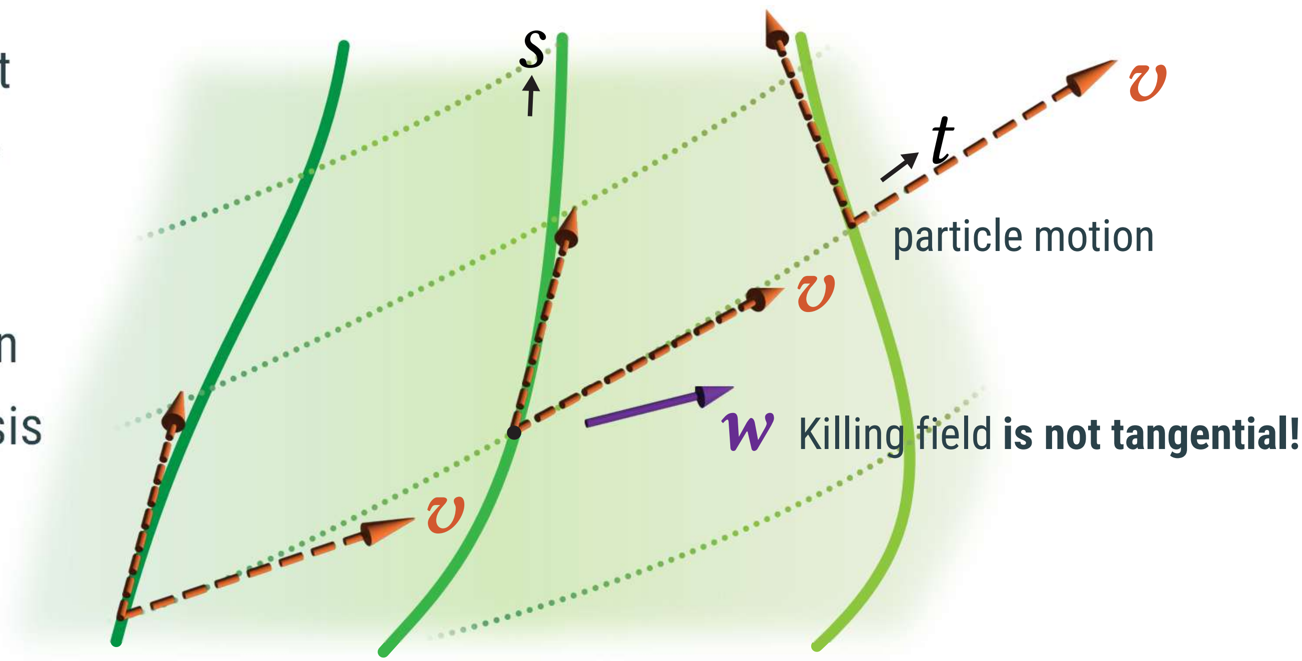
surface in spacetime

Quantify deformation

using spacetime basis

w does not work

v does not work

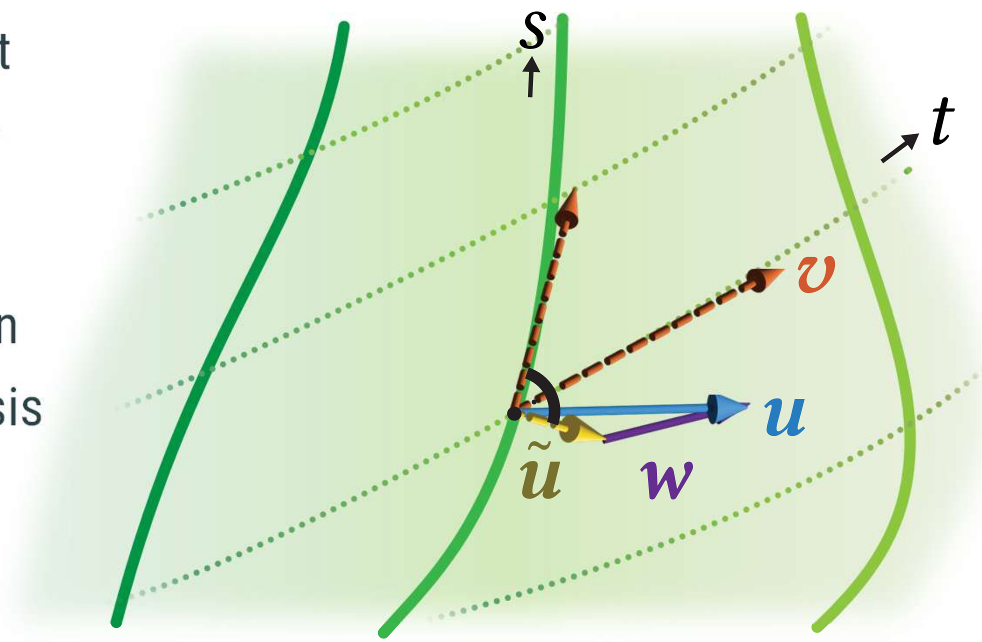


Spacetime Search: Minimally Deforming Core Line

Core line sweeps out *surface in spacetime*

Quantify deformation using spacetime basis

u *approximate* Killing field (tangential)



$\tilde{u}$ deformation field

w Killing field

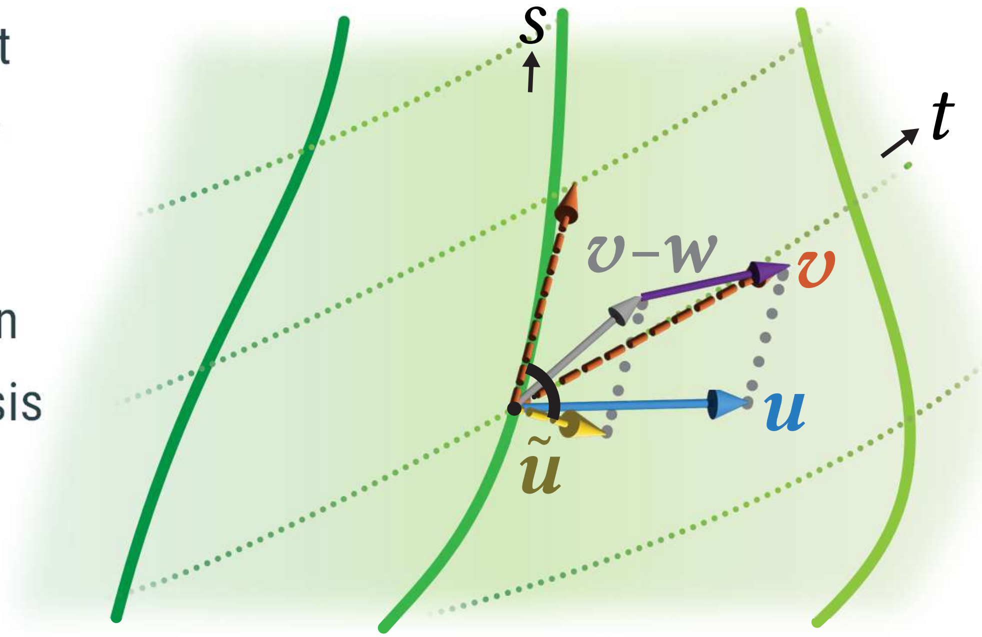
(both not tangential)

Spacetime Search: Minimally Deforming Core Line

Core line sweeps out *surface in spacetime*

Quantify deformation using spacetime basis

u *approximate*
Killing field
(tangential)



Minimize $\tilde{u}$

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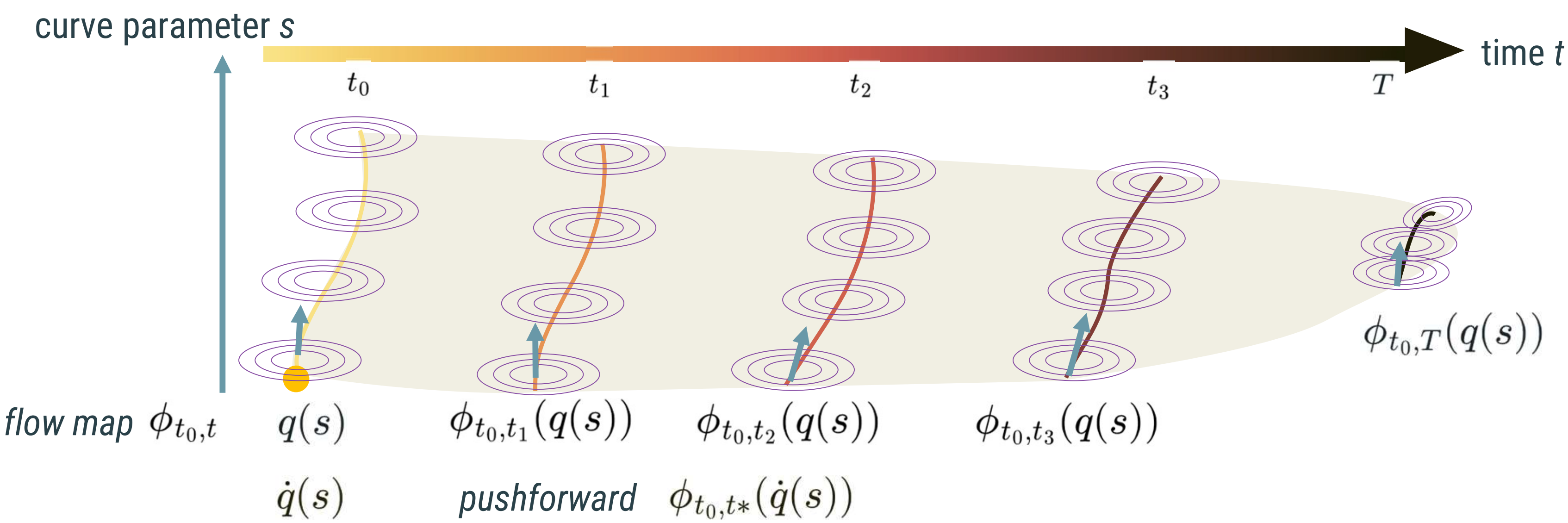
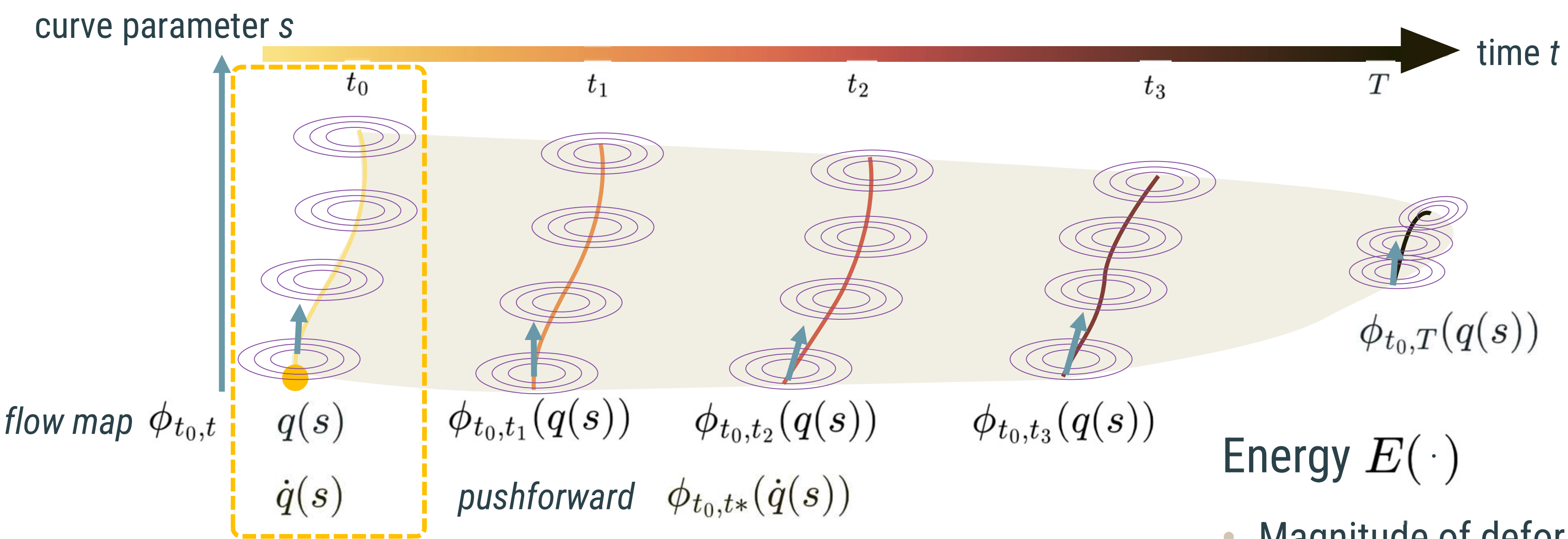
Key Idea #2

Define Lagrangian via Time Pre-Integration of Energy

Pre-Integrated Lagrangian: Curve gives Surface



Pre-Integrated Lagrangian: Curve gives Surface



$$L(q, \dot{q}, s) = \int_{t_0}^{T} E(\phi_{t_0,t}(q), \phi_{t_0,t*}(\dot{q}), t)\, dt$$

Energy $E(\cdot)$

- Magnitude of deformation $\tilde{\boldsymbol{u}}$
- Generic vortex criterion (e.g., IVD)
- Additional regularization

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Key Idea #3

Simplify Euler-Lagrange with Properties of our Lagrangian

Solving the Euler-Lagrange Equation

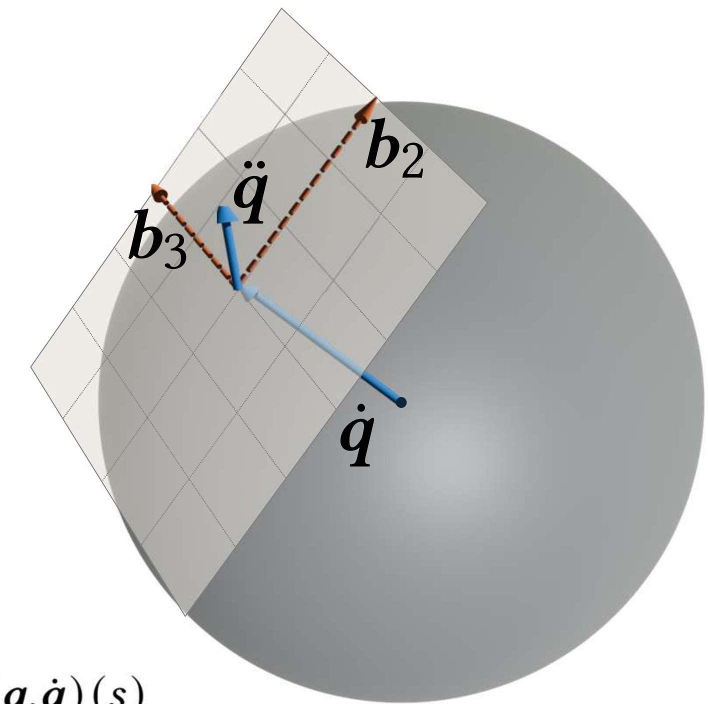
Lagrangian independent of $||\dot{\boldsymbol{q}}||$ + arc length constraint:

Solve for $\ddot{\boldsymbol{q}}$ in plane orthogonal to core line tangent $\dot{\boldsymbol{q}}$

Each step is simple 2x2 linear system solve

Compute 2x2 Hessian matrix in spherical coordinates

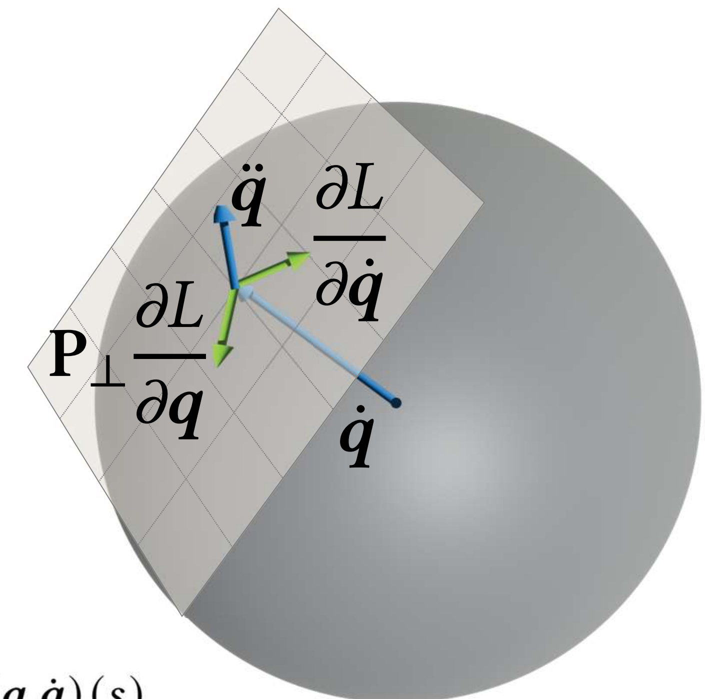
$$\ddot{\boldsymbol{q}}(s) = \left[(\boldsymbol{b}_2|\boldsymbol{b}_3) \left(\bar{\mathrm{H}} + 2\lambda(s)\,\mathrm{I}^{2\times 2}\right)^{-1} (\boldsymbol{b}_2|\boldsymbol{b}_3)^T \frac{\partial L}{\partial \boldsymbol{q}} \right]_{(\boldsymbol{q},\dot{\boldsymbol{q}})(s)}$$



Solving the Euler-Lagrange Equation

Lagrangian independent of $||\dot{\boldsymbol{q}}||$ + arc length constraint:

Solve for $\ddot{\boldsymbol{q}}$ *in plane orthogonal to core line tangent* $\dot{\boldsymbol{q}}$



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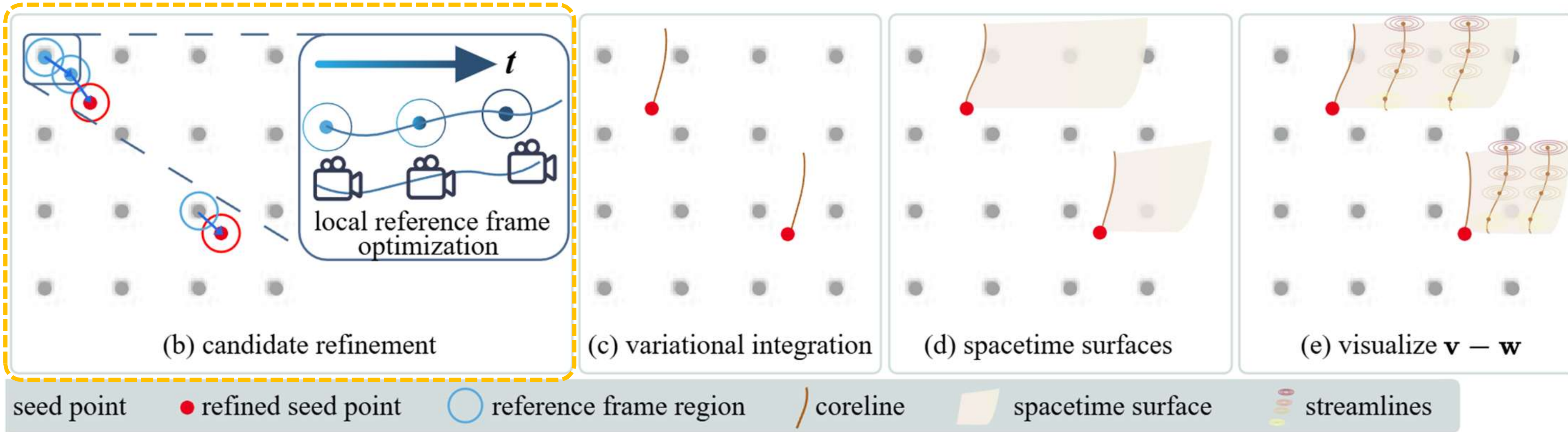
Pipeline

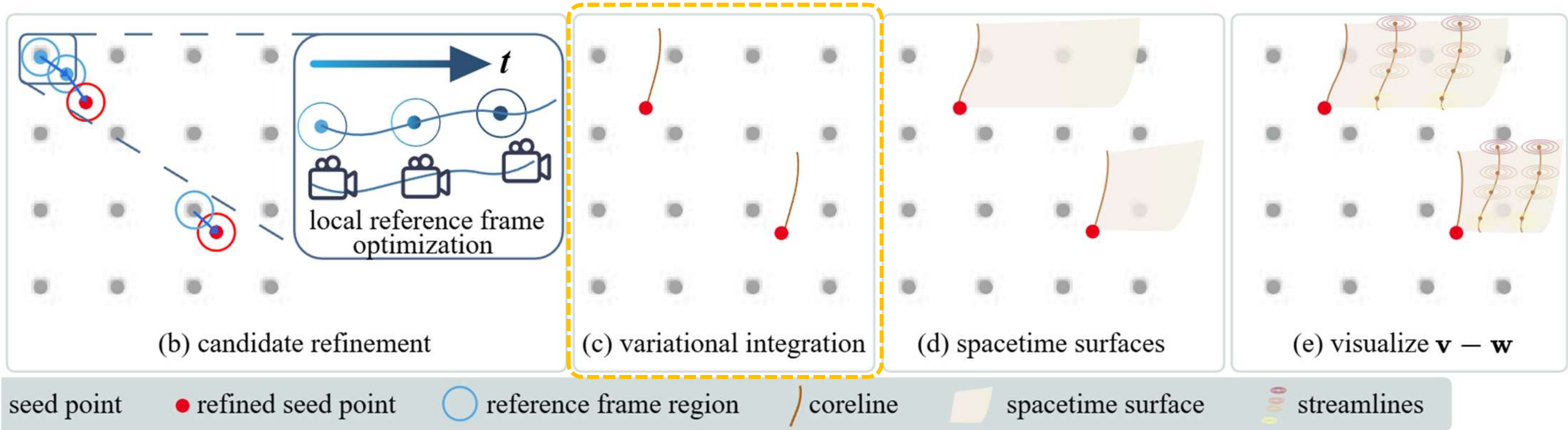
Bringing it All Together

Pipeline



Integrate Euler-Lagrange equation using Runge Kutta 4th order integration (RK4)

Pipeline



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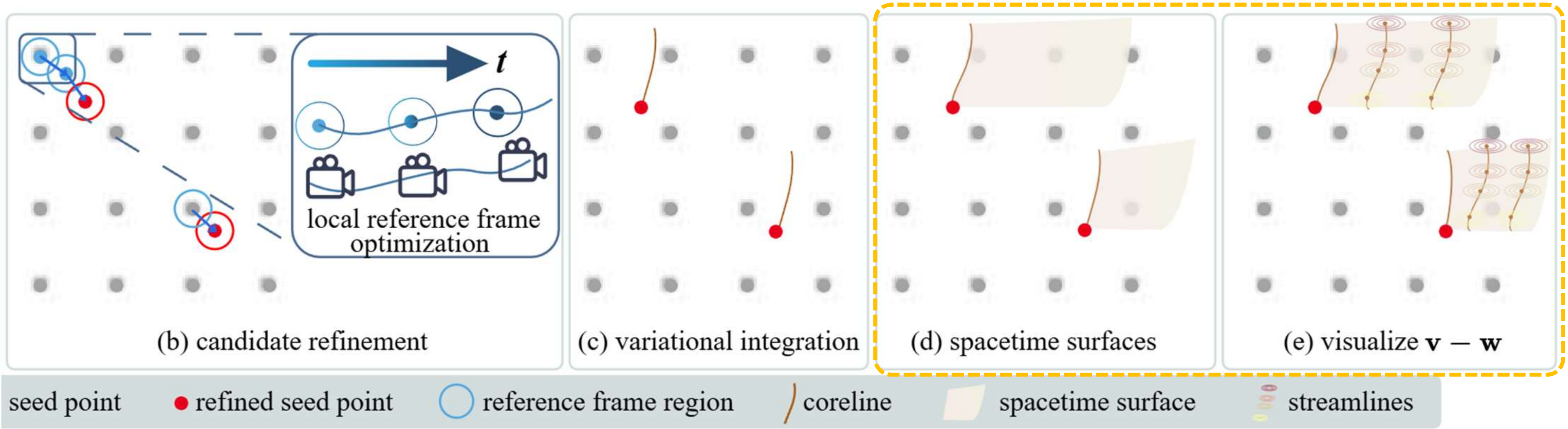
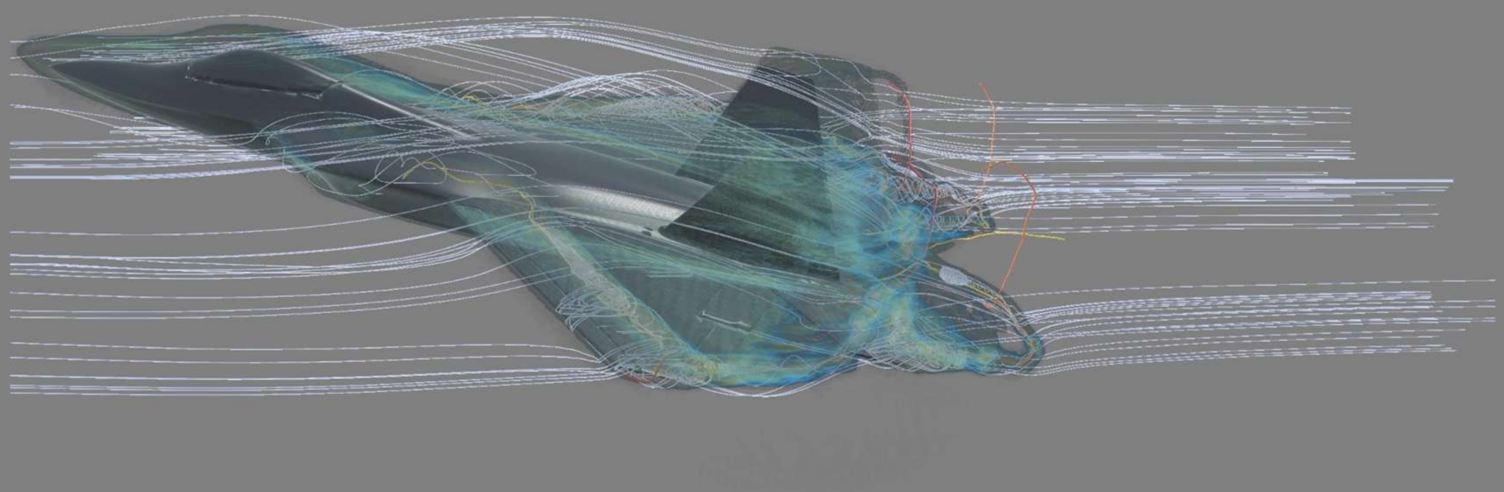
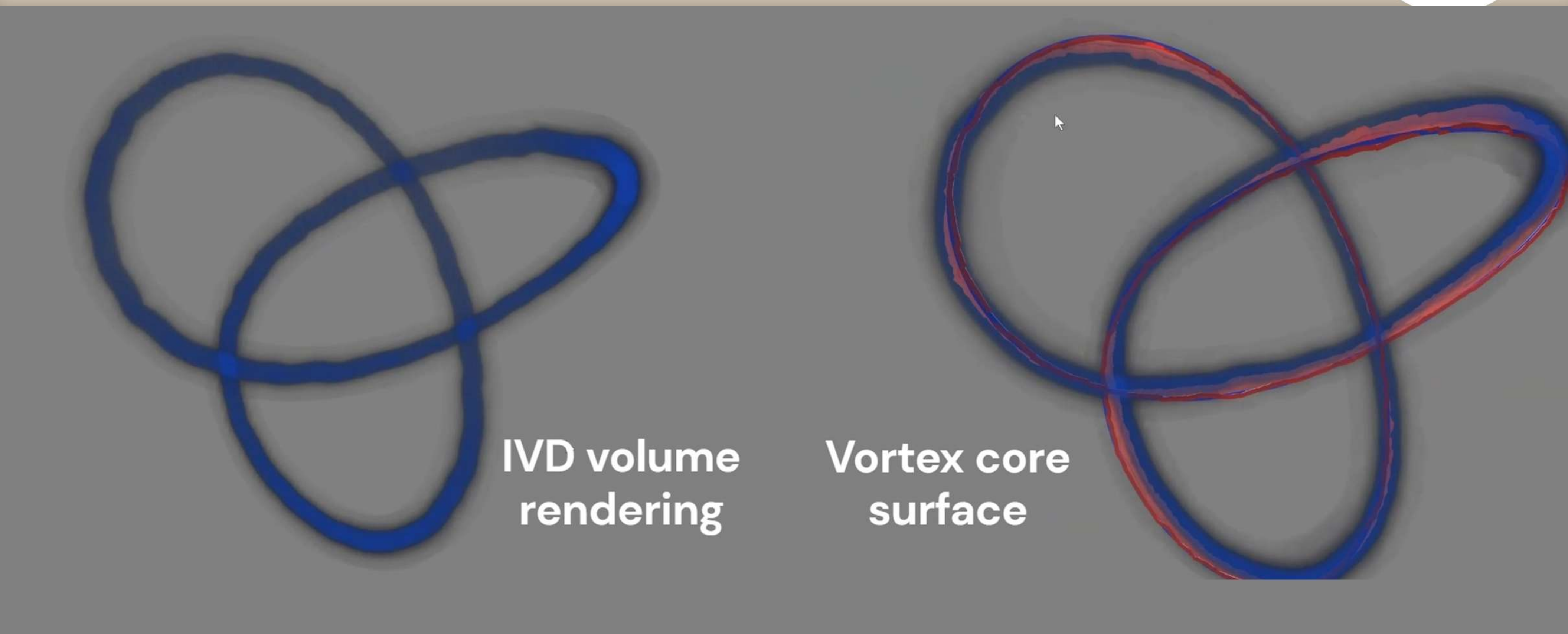


global reference frame → user → local reference frame

Result: Engineering Flow



Result: Trefoil Knot



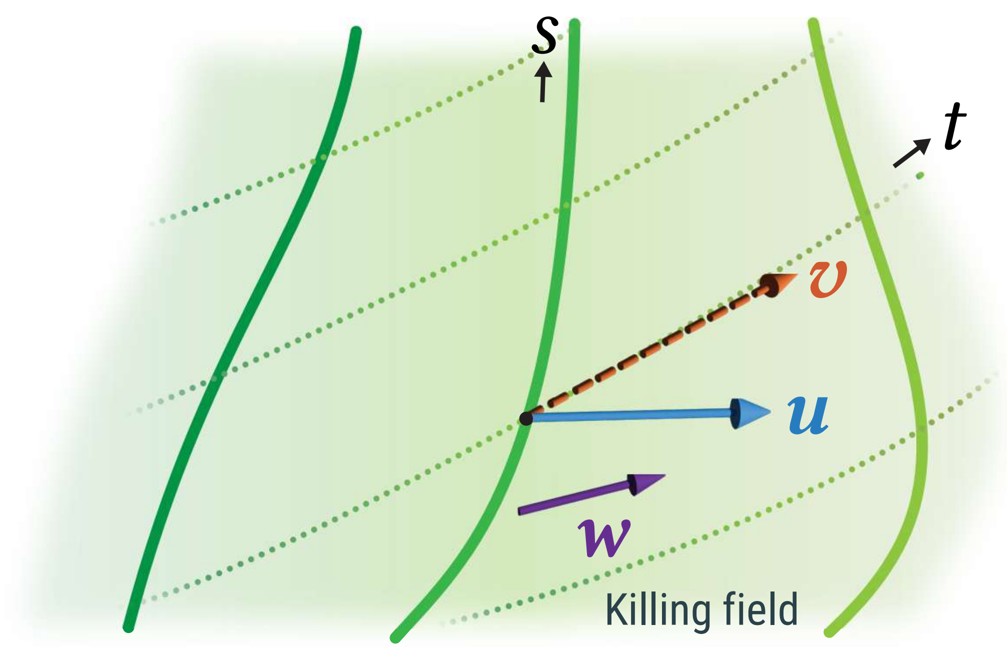
Conclusions and Limitations

Generalize steady case to unsteady case

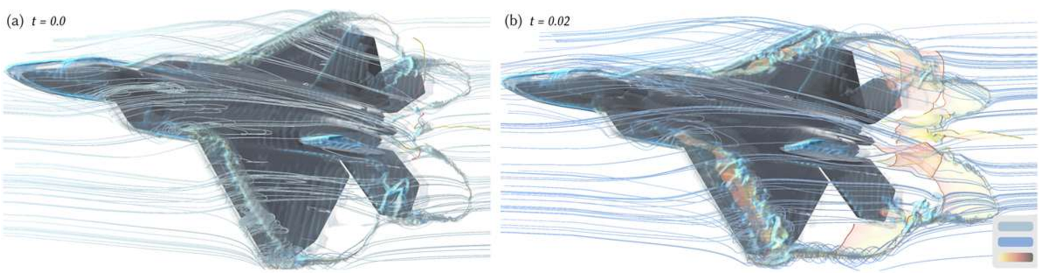
- Connect both: approximate Killing field
- Efficient: RK4 integration with 2x2 systems

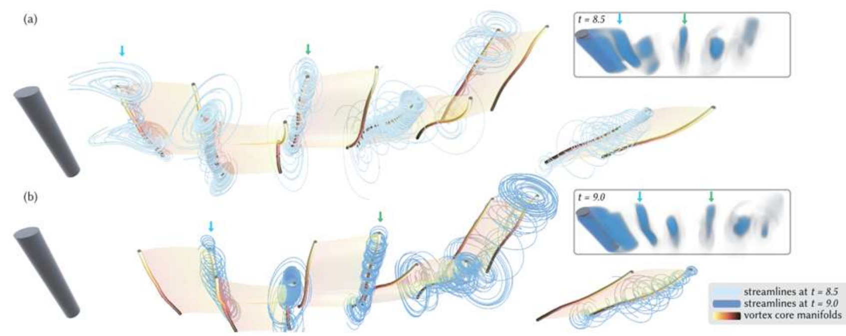
Limitations

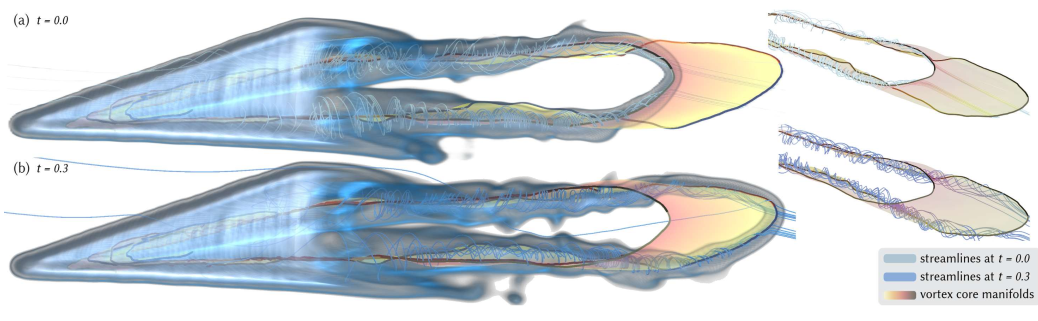
- Optimal Killing field computed for
  3D region of interest, not (unknown) core line
- Topology changes would require changing the set of particles comprising the vortex core



Thank you for listening!







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