Locally Adapted Reference Frame Fields using Moving Least Squares Supplemental Material

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1 Coreline Metrics for More Datasets

We give further λ parameter studies, for the datasets TURBULENCE2 (Table 1), TURBULENCE3 (Table 2), TURBULENCE4 (Table 3), TURBULENCE5 (Table 4), and CYLINDER (Table 5). For the dataset TURBULENCE5 (Table 4), the best overall metrics are obtained using $\lambda = 0.00625$, finding 2389 corelines compared to 1886 with the baseline. $\lambda = 0.0125$ yields 2627 corelines, but on average, the corelines are shorter and with higher mean segment error.

2 FTLE Study Images

Figs. 1 and 2 correspond to Sec. 4.1 (FTLE Guidance Field Computation) in the paper. We study the impact of the integration time T on the guided optimization by visualizing FTLE for multiple integration times for the dataset TURBULENCE1. We compute the FTLE field at time $t = 0$ for $k \in \{10, 100, 200\}$ (integration times), as illustrated in Fig. 1. Based on these results, for a given dataset, longer integration times produce well-defined ridges and valleys that can potentially improve the results. Despite this, we chose $k = 80$ to compare with the baseline, since the baseline produces satisfactory results for flows with moderate turbulence, and integration time requirements decrease for increasing turbulence levels, as seen in Fig. 2, where we fixed $k = 80$ and obtained FTLE fields for increasingly more turbulent data.

Quality Criteria	$\lambda = \mathbf{0.0125}$	$\lambda = 0.00625$	Baseline
m_{tac}	104.77	102.49	99.58
m_{mse}	1.87×10^{-4}	1.87×10^{-4}	1.80×10^{-4}
m_{epe}	5.49×10^{-3}	5.56×10^{-3}	5.57×10^{-3}
$\|C\|$	292	294	295
$\|\boldsymbol{v}_t^*\|_{total}$	5.98×10^7	6.20×10^7	6.48×10^7

Table 1: **Parameter study** for the TURBULENCE2 dataset.

Quality Criteria	$\lambda = 0.0125$	$\lambda = \mathbf{0.00625}$	Baseline
m_{tac}	92.83	98.86	90.92
m_{mse}	2.08×10^{-4}	1.88×10^{-4}	1.97×10^{-4}
m_{epe}	4.41×10^{-3}	4.75×10^{-3}	5.32×10^{-3}
$\|C\|$	427	394	416
$\|\boldsymbol{v}_t^*\|_{total}$	7.68×10^7	8.11×10^7	8.71×10^7

Table 2: **Parameter study** for the TURBULENCE3 dataset.

3 Additional LIC Visualizations

In Fig. 3, for the dataset TURBULENCE1 we compare the resulting LIC images and observed time derivatives using the baseline and our approach with high (1) and low (0.0125) λ values. The FTLE field color map in the first row covers the whole range [0, 11]. In contrast, the color map for the image in the second row covers the range [3,11] (threshold $\sigma \geq 3$) to highlight valleys (dark, low values) and exclude smaller ridges. In the third row (threshold FTLE and LICs), the first two close-ups in each column correspond to regions with ridges surrounding small valleys, where $\lambda = 0.0125$ produces subtly different results from the baseline. Instead, $\lambda = 1$ introduces small-scale features that do not match FTLE valleys, are not detected by the other methods, and the coreline metrics indicate that the corelines obtained from these small-scale features do not align with the pathlines. In regions with larger valleys (third and fourth close-ups in each column), the baseline and ours with $\lambda = 0.0125$ produce equivalent results. In the last row, we notice that $\|\boldsymbol{v}_t^*\|$ for $\lambda = 1$ and the baseline are indistinguishable and its valleys match those in FTLE, while $\lambda = 1$ produces an overall smaller observed time partial with larger valleys.

Quality Criteria	$\lambda = 0.0125$	$\lambda = \mathbf{0.00625}$	Baseline
m_{tac}	91.78	91.27	87.44
m_{mse}	2.24×10^{-4}	2.27×10^{-4}	2.34×10^{-4}
m_{epe}	5.14×10^{-3}	5.29×10^{-3}	5.84×10^{-3}
$\|C\|$	667	648	641
$\|\boldsymbol{v}_t^*\|_{total}$	1.09×10^8	1.18×10^8	1.33×10^8

Table 3: **Parameter study** for the TURBULENCE4 dataset.

Quality Criteria	$\lambda = 0.0125$	$\lambda = \mathbf{0.00625}$	Baseline
m_{tac}	49.90	51.38	50.96
m_{mse}	2.35×10^{-4}	2.21×10^{-4}	2.36×10^{-4}
m_{epe}	3.27×10^{-3}	3.55×10^{-3}	4.45×10^{-3}
$\|C\|$	2627	2389	1886
$\|\boldsymbol{v}_t^*\|_{total}$	1.28×10^8	1.48×10^8	2.11×10^8

Table 4: **Parameter study** for the TURBULENCE5 dataset.

4 Kármán Vortex Street

During our experiments, we evaluated the baseline and our method using the CYLINDER dataset. For this laminar flow where vortices move mostly in the same direction, both methods provide equivalent results. The coreline metrics are in Table 5. In Fig. 4, we show $\|\boldsymbol{v}_t^*\|$ and LIC with color-coded vorticity in the optimal frame.

5 Compactly Supported Weighting Functions

In addition to the Gaussian and exponential weighting functions, we tested some of Wendland's compactly supported functions [2] of minimal degree. We tested $\phi_{3,0}$ (C^0), $\phi_{3,1}$ (C^2) and $\phi_{3,2}$ (C^4). We conducted a parameter study for λ, and chose $\lambda = 0.03$. The coreline metrics for the TURBULENCE5 dataset with Wendland's weighting functions using $\lambda = 0.03$ are in Table 6. The results are similar to those obtained using the Gaussian weighting function and $\lambda = 0.00625$, especially for $\phi_{3,1}$, which produces equivalent neighborhood weights and corelines.

6 Additional Coreline Comparisons

In Fig. 5, we show the coreline comparison between the baseline and our method for datasets TURBULENCE2 and TURBULENCE4.

References

[1] T. Günther, M. Gross, and H. Theisel. Generic objective vortices for flow visualization. *ACM Transactions on Graphics*, 36(4):141:1–141:11, 2017. doi: 10.1145/3072959.3073684 4

[2] H. Wendland. *Scattered Data Approximation*. Cambridge University Press, 2010. 1, 2

Quality Criteria	$\lambda = 1$	$\lambda = 0.03$	$\lambda = \mathbf{0.02}$	$\lambda = 0.01$	$\lambda = 0.00625$	Baseline
m_{tac}	67.60	80.90	80.87	78.26	80.77	80.73
m_{mse}	1.13×10^{-3}	8.43×10^{-4}	8.46×10^{-4}	1.01×10^{-3}	8.50×10^{-4}	8.53×10^{-4}
m_{epe}	9.41×10^{-3}	2.73×10^{-2}	2.72×10^{-2}	2.65×10^{-2}	2.71×10^{-2}	2.66×10^{-2}
$\|C\|$	35	30	30	31	30	30
$\|\boldsymbol{v}_t^*\|_{total}$	1.96×10^5	2.65×10^5	2.69×10^5	2.73×10^5	2.75×10^5	2.79×10^5

Table 5: **Parameter study** for the CYLINDER dataset.

Quality Criteria	Wendland C^0	Wendland C^2	Wendland C^4
m_{tac}	53.20	50.80	50.03
m_{mse}	2.27×10^{-4}	2.25×10^{-4}	2.29×10^{-4}
m_{epe}	3.73×10^{-3}	3.53×10^{-3}	3.46×10^{-3}
$\|C\|$	2231	2415	2525
$\|\boldsymbol{v}_t^*\|_{total}$	1.58×10^8	1.49×10^8	1.40×10^8

Table 6: **Parameter study** for the TURBULENCE5 dataset using Wendland's compactly supported functions of minimal degree [2]

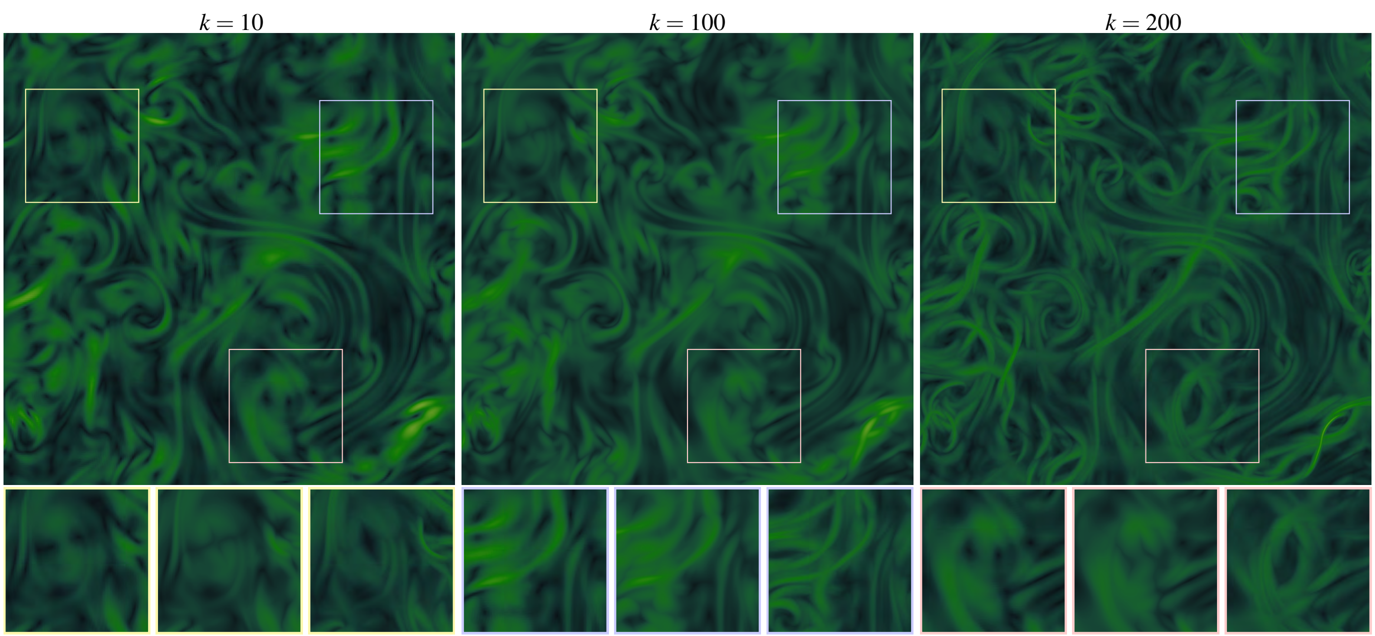

Fig. 1: **FTLE fields for TURBULENCE1** at time $t = 0$, increasing k (integration time) values. Longer integration times produce sharper guidance fields.

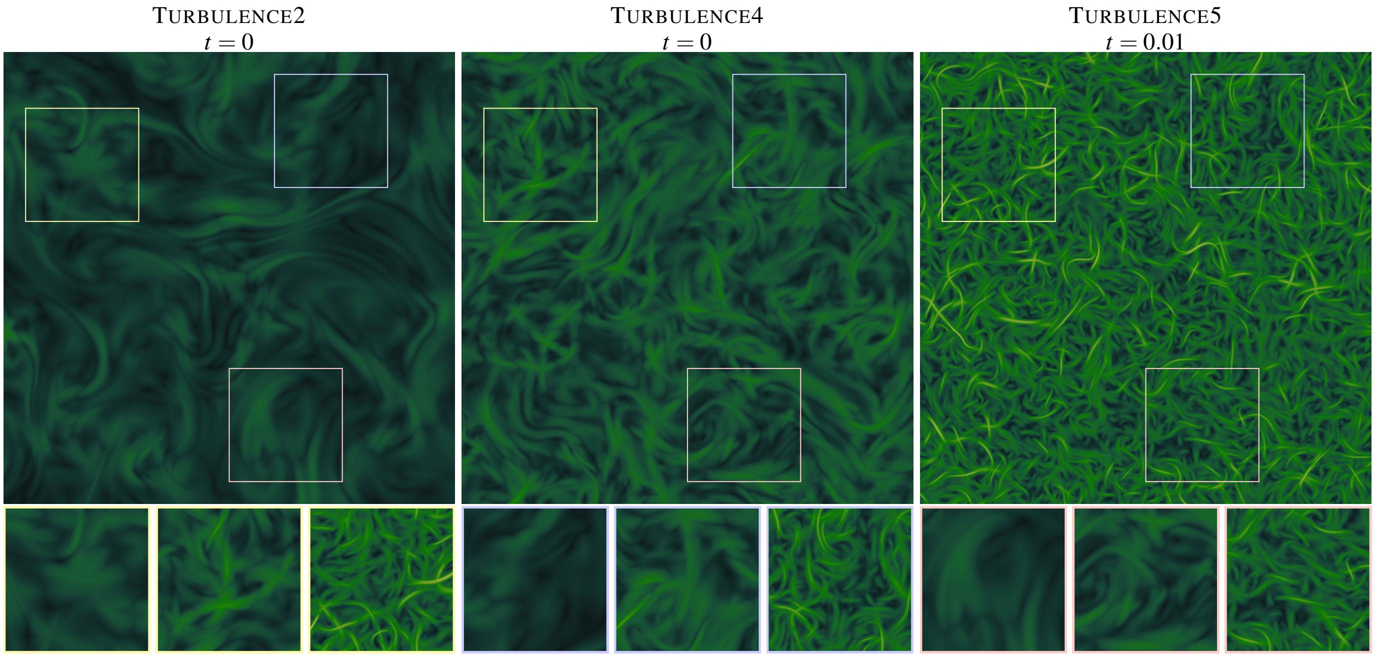

Fig. 2: FTLE fields with $k = 80$ across different turbulence levels. Flow separatrices are sharper at higher turbulence using the same integration time.

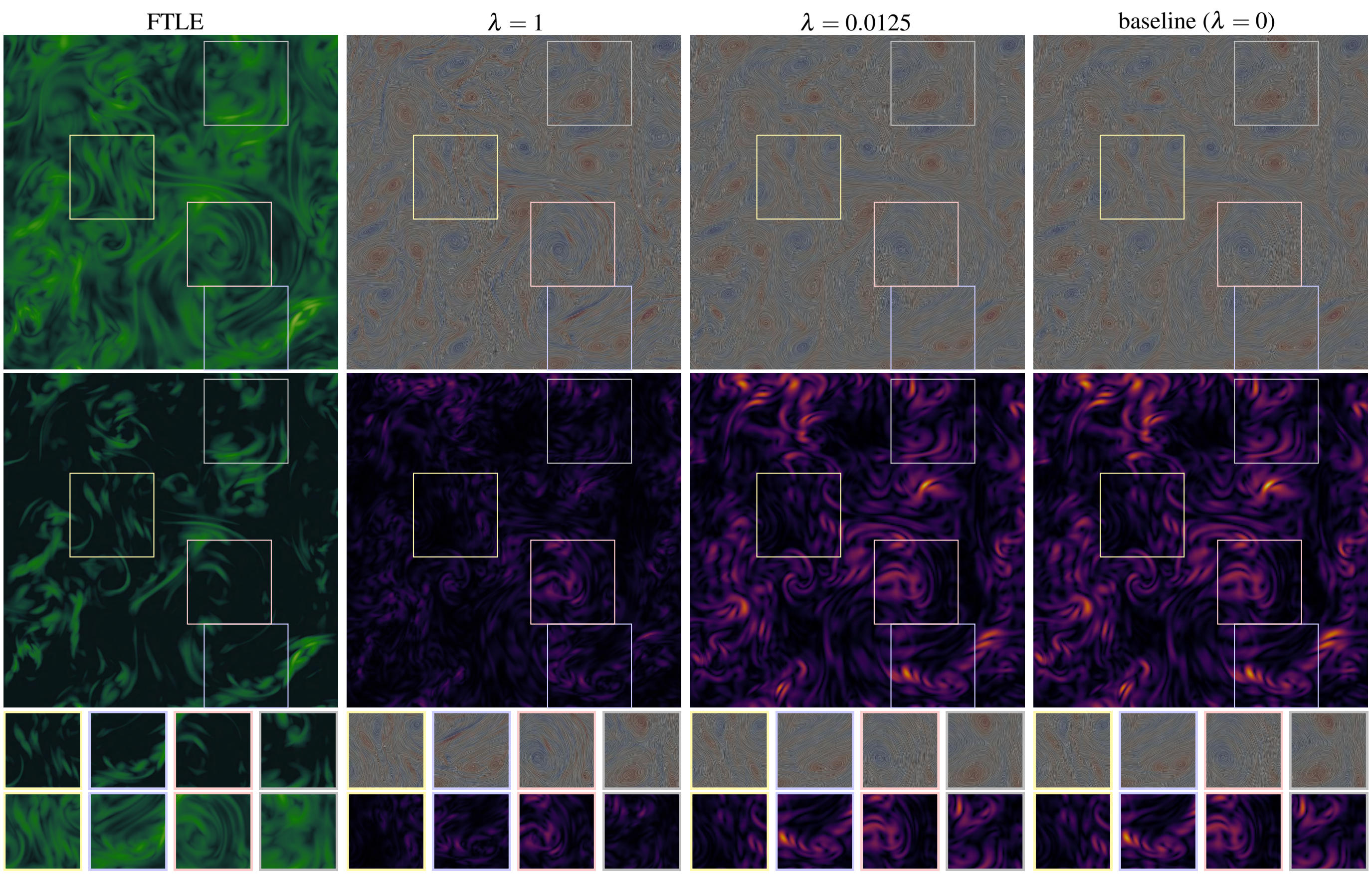

Fig. 3: FTLE, LIC and $\|\boldsymbol{v}_t^*\|$ of TURBULENCE1 at time $t = 0$. First row: FTLE field and LIC images with color-coded vorticity. Second row: FTLE field with threshold $\sigma \geq 3$ and $\|\boldsymbol{v}_t^*\|$. Third row: FTLE (threshold) and LIC images. Fourth row: FTLE and $\|\boldsymbol{v}_t^*\|$.

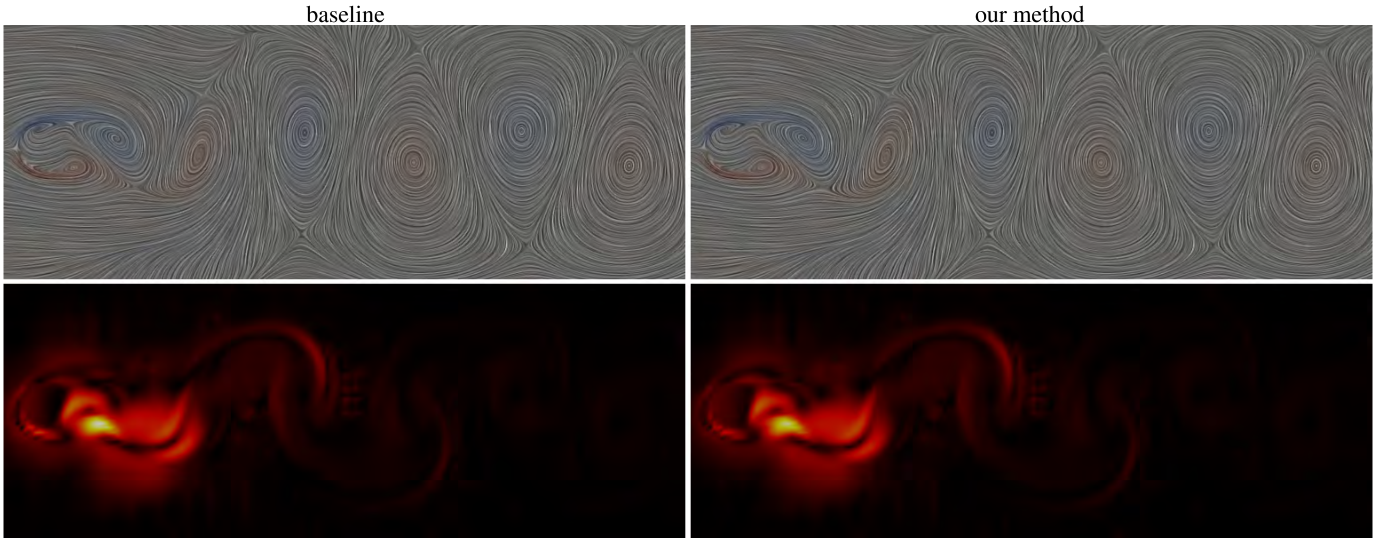

Fig. 4: Close-up of recently shed vortices in the CYLINDER dataset. For this laminar flow, the results are almost identical. The top images show the flow in the optimal reference frame using LIC with color-coded vorticity. The bottom images show the observed time partial $\|\boldsymbol{v}_t^*\|$.

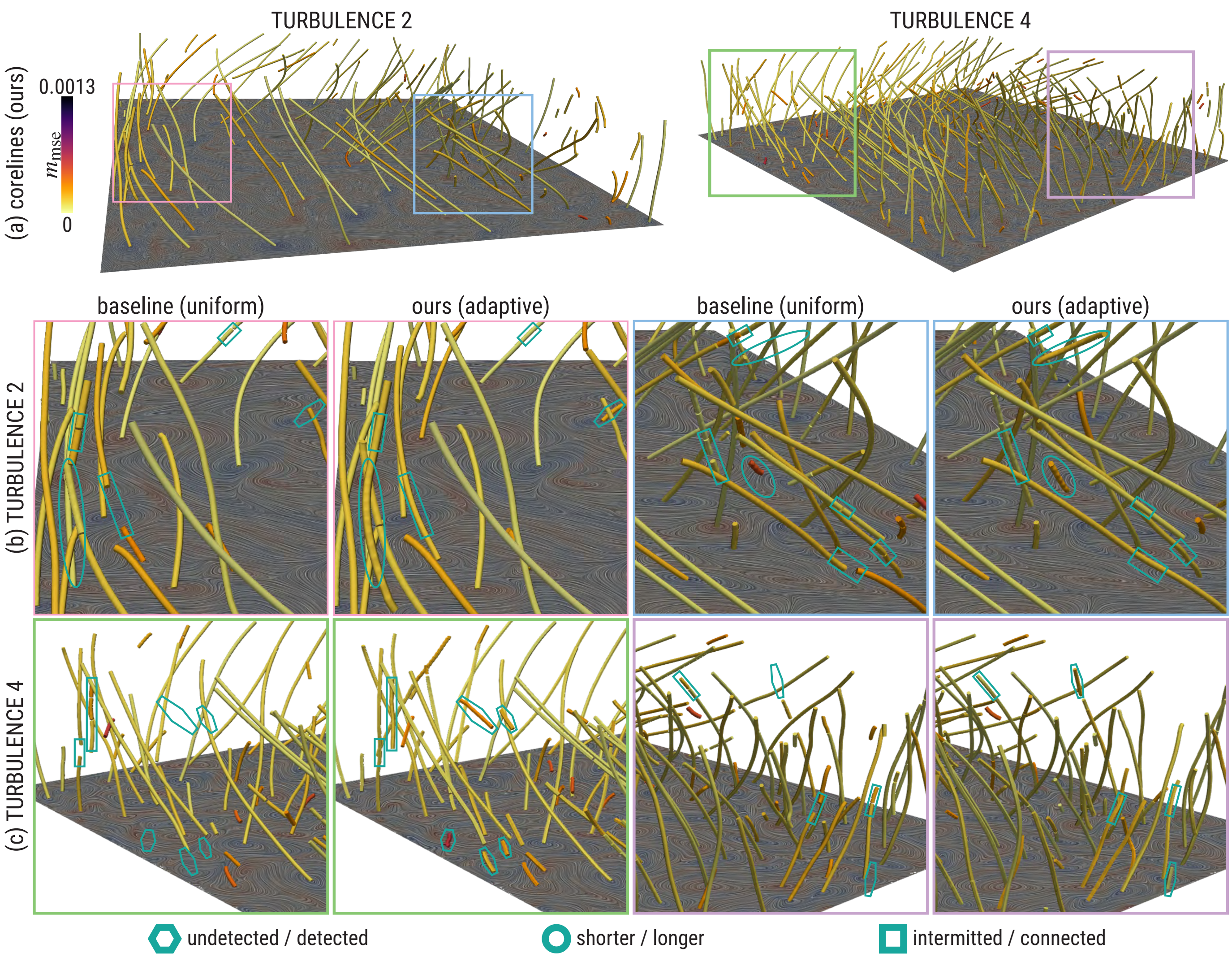

Fig. 5: **Coreline comparisons** for TURBULENCE2 and TURBULENCE4. (a) corelines (ours) from TURBULENCE2 (left), and TURBULENCE4 (right), where the coreline color encodes the mean segment error per coreline. (b) TURBULENCE2 corelines zoom in, where the baseline [1] results are on the left and our results are on the right for each pair of insets. (c) TURBULENCE4 zoom ins.