Time-Dependent Flow seen through Approximate Observer Killing Fields

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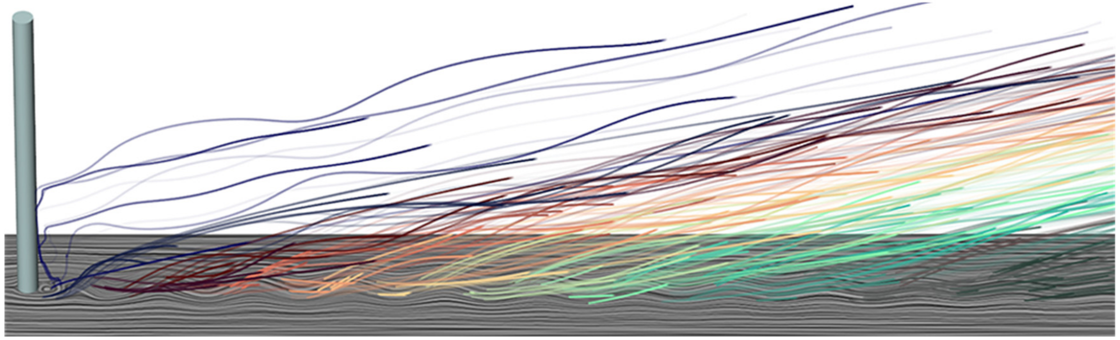
Flow Visualization is Observer-Relative

Steady vs. unsteady flow

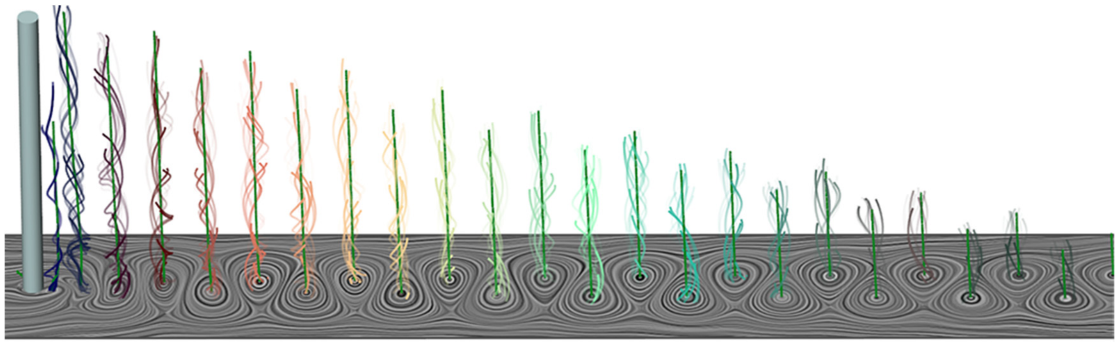
Flow features (vortices, …)

One frame of reference cannot depict all features

Continuous observer field adapted to input flow



vortex street in laboratory frame

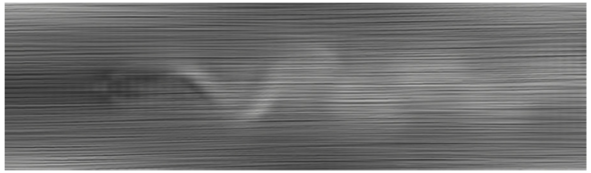


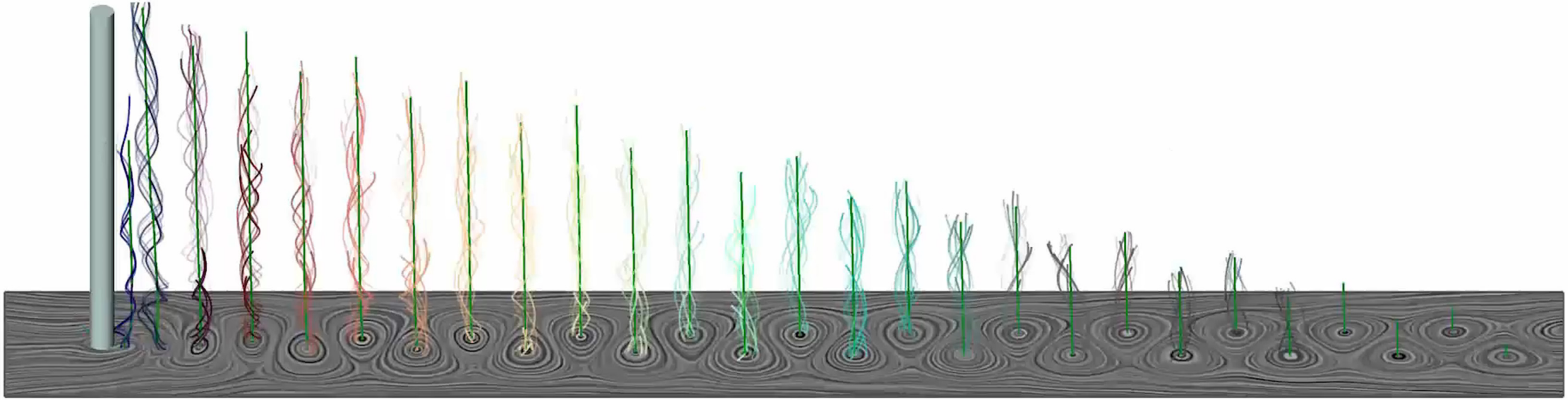
relative to approximate observer Killing field

Mathematical Framework

Observer field from global optimization

- Carefully define desired differential properties
- Minimize them over all of space-time





Observers and Killing Vector Fields

Wilhelm Killing (1847-1923)

Observer Definition

Space that is in relative rigid motion

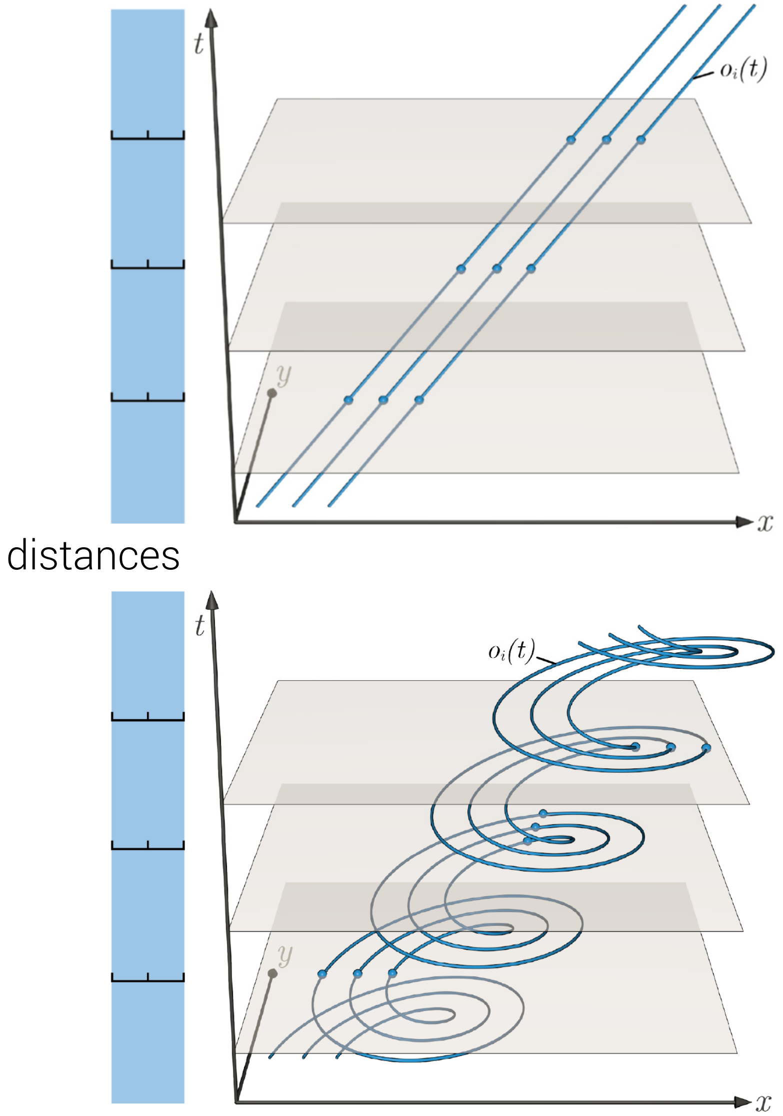
- Relative to what? Another observer

Points stay at constant distance

- Motion described by isometries
- Point trajectories are **world lines**

Isometries of Euclidean space

- Time-dependent translations
- Time-dependent rotations



Describing Rigid Observer Motion

Euclidean isometries: time-dependent translations and rotations

$$c(t) \qquad\qquad \mathbf{Q}(t)$$

Easier to work with derivatives (infinitesimal isometries)

$$\mathbf{w}(t) := \dot{c}(t) \qquad\qquad \mathbf{\Omega}(t) := \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T$$

Describing Rigid Observer Motion

Euclidean isometries: time-dependent translations and rotations

$$c(t) \qquad \boxed{\mathbf{Q}(t) \qquad \textit{Lie group} \quad SO(n)}$$

Easier to work with derivatives (infinitesimal isometries)

(linear!)

$$\mathbf{w}(t) := \dot{c}(t) \qquad \boxed{\mathbf{\Omega}(t) := \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T \qquad \textit{Lie algebra } \mathfrak{so}(n)}$$

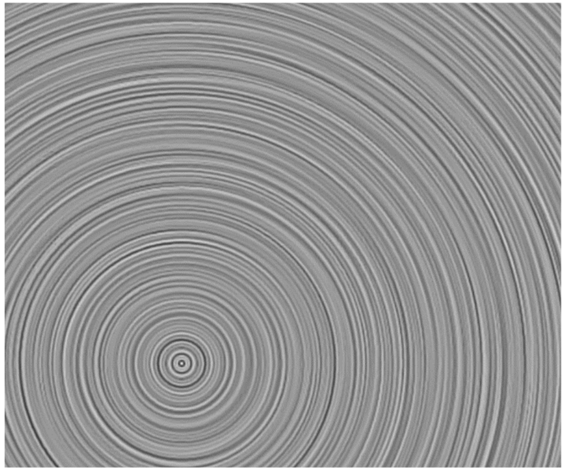
Killing fields are infinitesimal isometries (Lie algebra of isometry group)

$$\mathbf{u}(x,t) := \mathbf{w}(t) + \mathbf{\Omega}(t)\big(x - o(t)\big)$$

Rigid Motion as Killing Vector Field

Killing field (time-dependent)

$$\mathbf{u}(x,t) = \mathbf{w}(t) + \mathbf{\Omega}(t)(x - o(t))$$



$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

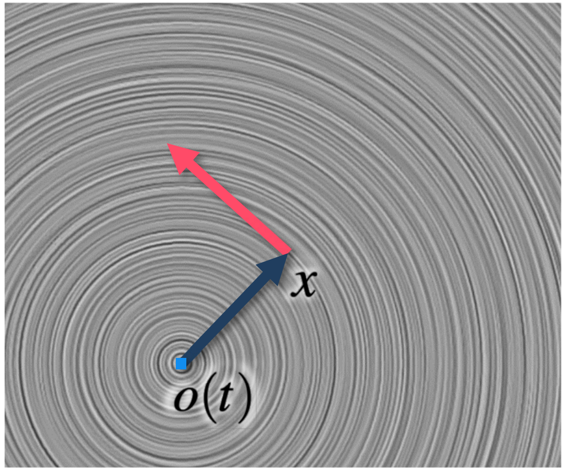
Killing's equation:

$$\nabla \mathbf{u} + (\nabla \mathbf{u})^T = 0$$

Evaluating Killing Vector Fields

Translation corresponds to velocity of *some* point $\boldsymbol{o(t)}$

$$\mathbf{u}(x,t) = \mathbf{u}(o(t),t) + \boxed{\mathbf{\Omega}(t)\boxed{(x-o(t))}}$$



$$\mathbf{w}(t) = \mathbf{u}(o(t),t)$$

Infinitesimal translation of $\boldsymbol{o(t)}$

Evaluating Killing Vector Fields

Translation corresponds to velocity of *some* point $o(t)$

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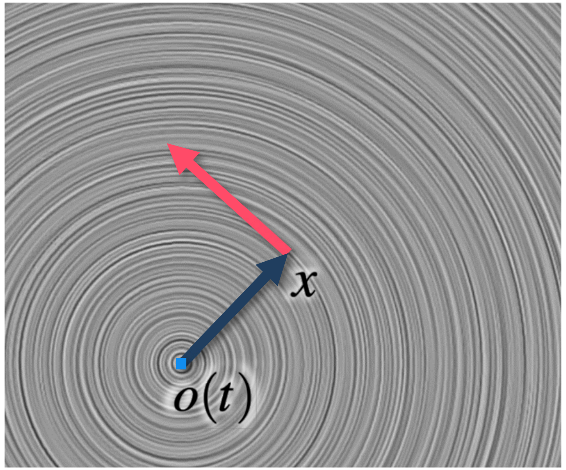

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Killing Fields are Independent of "Origin"

Choice of $o(t)$ is arbitrary (well-known property of Killing fields)

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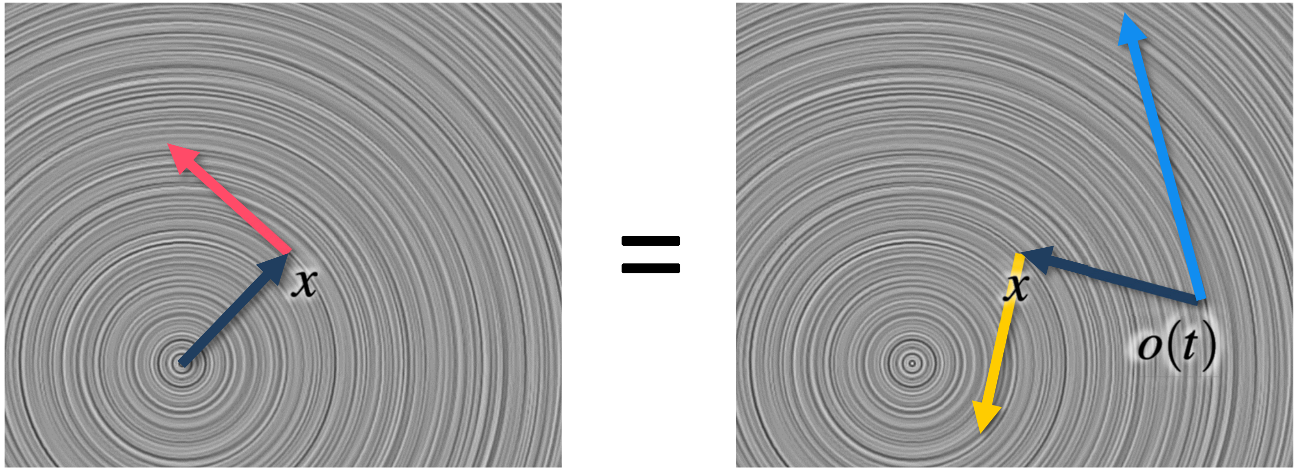

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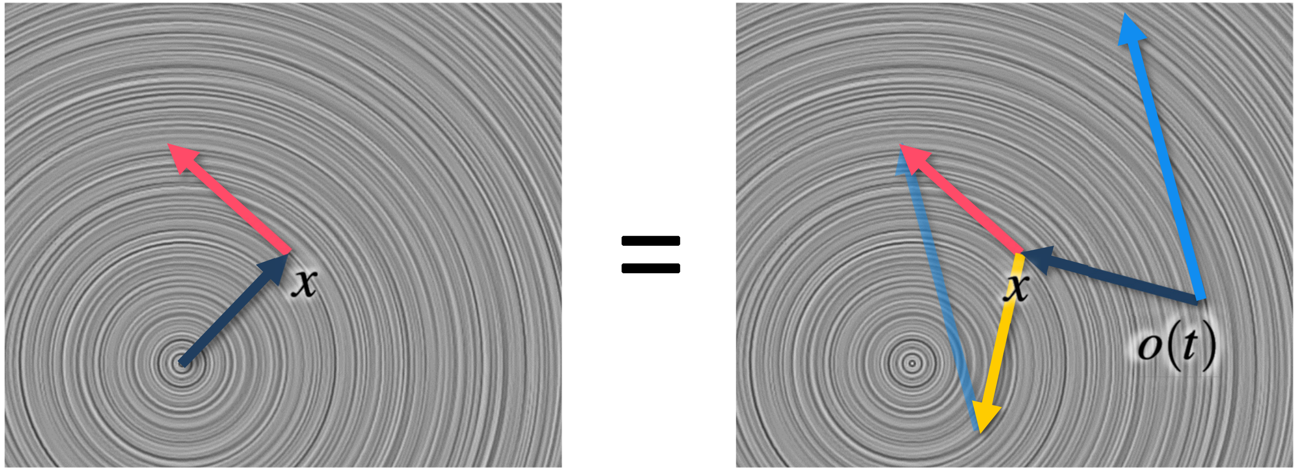
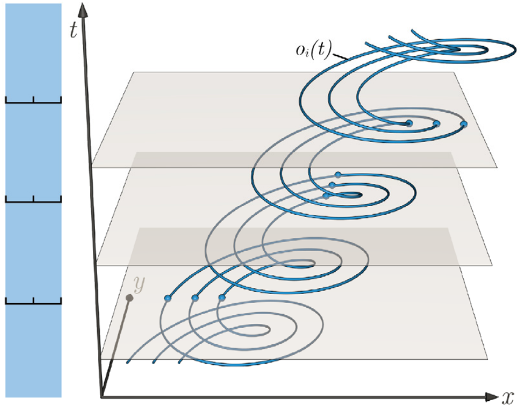


Observer Fields: One Observer

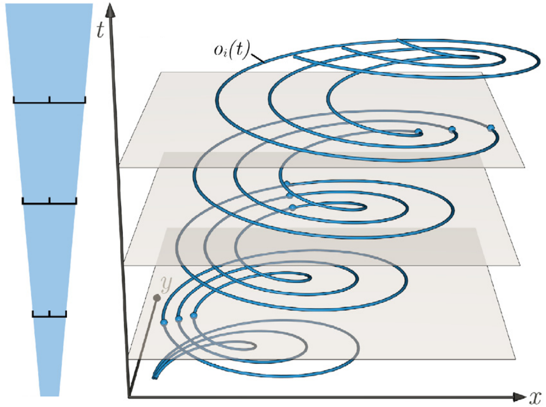


One Killing field describes one observer $\boldsymbol{i}$

$$\mathbf{u}_i\left(x,t\right) = \boxed{\mathbf{u}_i(o_i(t),t)} + \mathbf{\Omega}_i(t)\left(x - o_i(t)\right)$$

$\mathbf{u}_i\left(x,t\right)$ is a Killing field

Observer Fields: Multiple Observers



Continuous observer field $\mathbf{u}$

$$\mathbf{u}_i(x,t) := \boxed{\mathbf{u}(o_i(t),t)} + \mathbf{\Omega}_i(t)(x - o_i(t))$$

World lines $o_i(t)$: integral curves of $\mathbf{u}$ and $\mathbf{\Omega}_i(t) := \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right)(o_i(t),t)$

$\mathbf{u}_i(x,t)$ is a Killing field

$\mathbf{u}(x,t)$ is arbitrary

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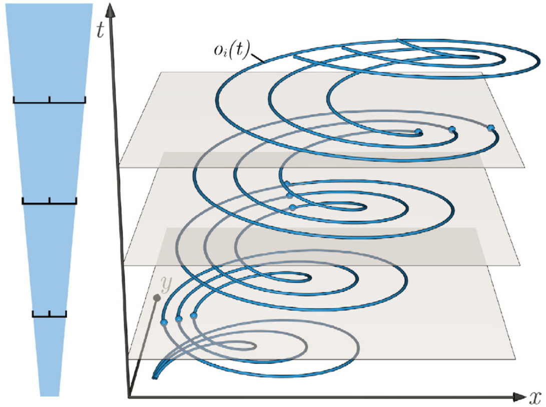

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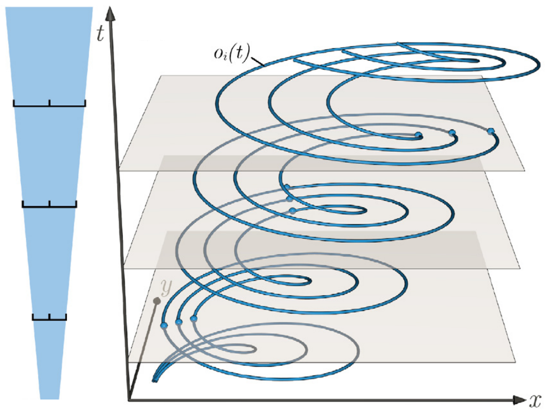
Approximate Observer Killing Fields



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Each $\mathbf{u}_i(x,t)$ is a Killing field (one per observer)

Choose $\mathbf{u}(x,t)$ to be an approximate Killing field:

min. *Killing energy* $K\mathbf{u} := \nabla\mathbf{u} + (\nabla\mathbf{u})^T$

Time Derivatives

Observed Time Derivative

Original observer:

No relative observer motion

$$\boxed{\mathbf{u} = 0}$$

$$\frac{\mathscr{D}}{\mathscr{D}t}\mathbf{v}_{\mathbf{u}} = \frac{\partial \mathbf{v}}{\partial t}$$

Observer field-relative time derivative

Field $\mathbf{v}_{\mathbf{u}} = \mathbf{v} - \mathbf{u}$ as perceived by the observer field $\mathbf{u}$

Observed Time Derivative

Galilean observer:

Constant-velocity translation

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} = 0 \qquad \mathbf{\Omega} = 0}$$

$$\frac{\mathscr{D}}{\mathscr{D}t}\mathbf{v}_{\mathbf{u}} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v}\left(\mathbf{u}\right)$$

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Observer field-relative time derivative

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Observed Time Derivative

Rigid motion observer:

Exact Killing field

$$\boxed{\nabla \mathbf{u} = \mathbf{\Omega}}$$

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Observer field-relative time derivative

Field $\mathbf{v}_{\mathbf{u}} = \mathbf{v} - \mathbf{u}$ as perceived by the observer field $\mathbf{u}$

Observed Time Derivative

General observer:

Arbitrary observer field

$$\boxed{\frac{\mathscr{D}}{\mathscr{D}t}\mathbf{v}_{\mathbf{u}} = \frac{\partial \mathbf{v}}{\partial t} - \frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{v}\,(\mathbf{u}) - \nabla \mathbf{u}\,(\mathbf{v})}$$

Observer field-relative time derivative

Field $\mathbf{v}_{\mathbf{u}} = \mathbf{v} - \mathbf{u}$ as perceived by the observer field $\mathbf{u}$

Observed Time Derivative is a Lie Derivative

General observer:

Arbitrary observer field

$$\boxed{\frac{\mathscr{D}}{\mathscr{D}t}\mathbf{v}_{\mathbf{u}} = L_{\mathbf{u}}(\mathbf{v}_{\mathbf{u}})} = \frac{\partial \mathbf{v}}{\partial t} - \frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{v}\,(\mathbf{u}) - \nabla \mathbf{u}\,(\mathbf{v})$$

Time-dependent **Lie derivative** of $\mathbf{v}_{\mathbf{u}} = \mathbf{v} - \mathbf{u}$ with respect to $\mathbf{u}$

Important: Lie derivatives of objective vector fields are objective

Optimization

Approximate Observer Killing Fields

Global L_2 minimization over space and time (sparse linear system)

$$\min_{\mathbf{u}} \int_{\tau,\xi} \left(E_K + \lambda D_t + \mu R\right)(\mathbf{u},\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau$$

Goals: approximately Killing, small observed time derivative

$$E_K(\mathbf{u},\xi,\tau) := \frac{1}{2}\,\|K\mathbf{u}(\xi,\tau)\|_F^2 \qquad D_t(\mathbf{u},\xi,\tau) := \frac{1}{2}\left\|\frac{\mathscr{D}}{\mathscr{D}t}\mathbf{v}_{\mathbf{u}}(\xi,\tau)\right\|_2^2$$

Approximate Observer Killing Fields

Global L_2 minimization over space and time (sparse linear system)

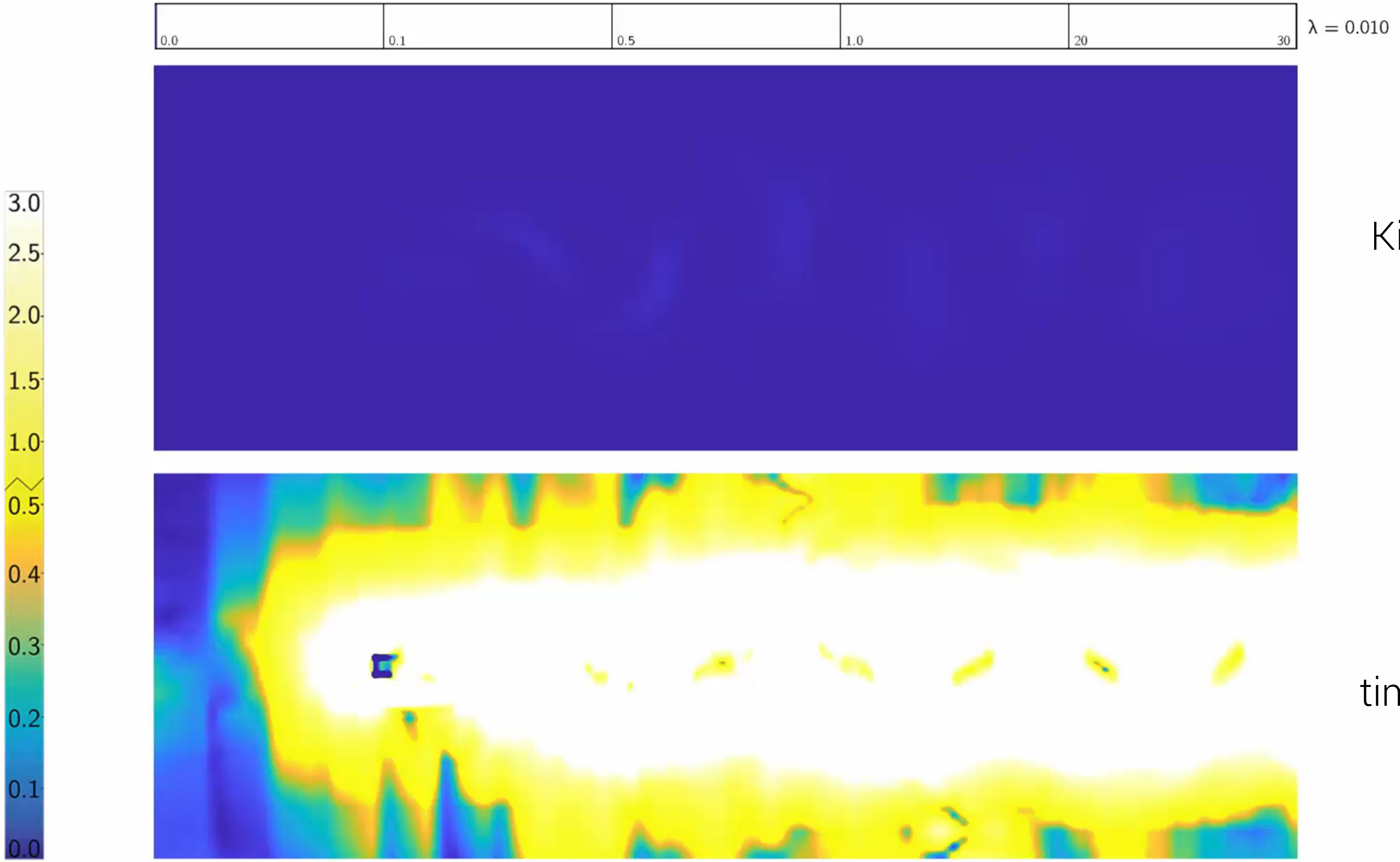
$$\min_{\mathbf{u}} \int_{\tau,\xi} \left(E_K + \lambda\, D_t + \mu\, R\right)(\mathbf{u}, \xi, \tau)\; \mathrm{d}\xi\; \mathrm{d}\tau$$

Goals: approximately Killing, small observed time derivative

$$E_K(\mathbf{u}, \xi, \tau) := \frac{1}{2} \| K\mathbf{u}(\xi, \tau) \|_F^2 \qquad D_t(\mathbf{u}, \xi, \tau) := \frac{1}{2} \left\| \frac{\mathscr{D}}{\mathscr{D}t} \mathbf{v}_{\mathbf{u}}(\xi, \tau) \right\|_2^2$$

regularizer $R(\mathbf{u}, \xi, \tau) := \frac{1}{2} \| \mathbf{v}_{\mathbf{u}}(\xi, \tau) \|_2^2$

Approximate Observer Killing Fields

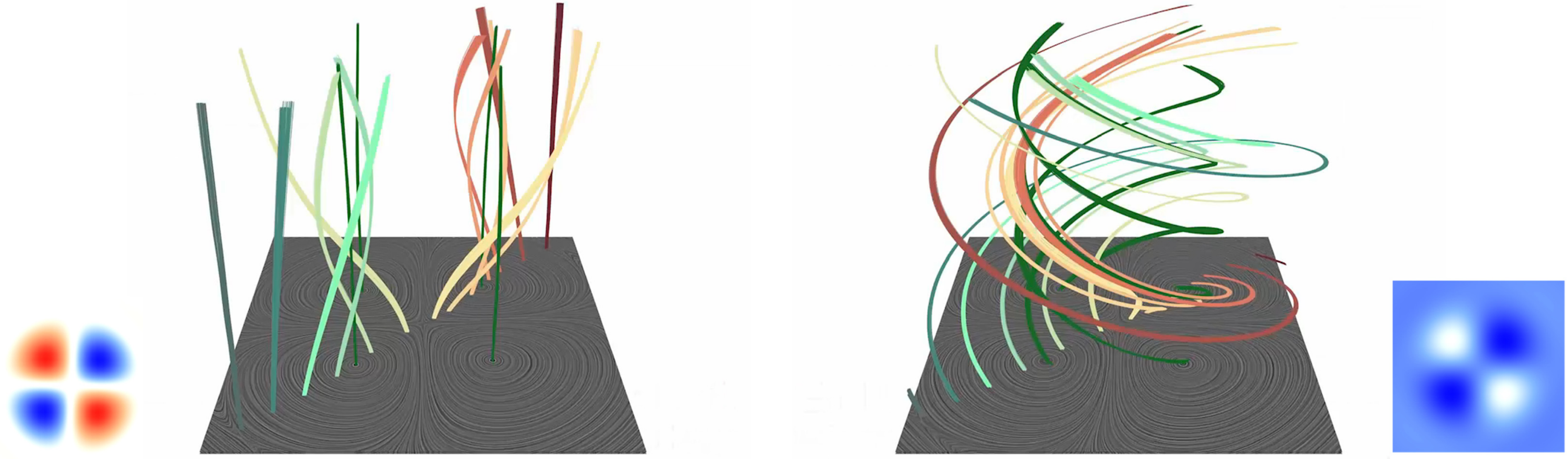


Killing energy

observed
time derivative

Visualization

Visualization Using Observer Fields



vorticity

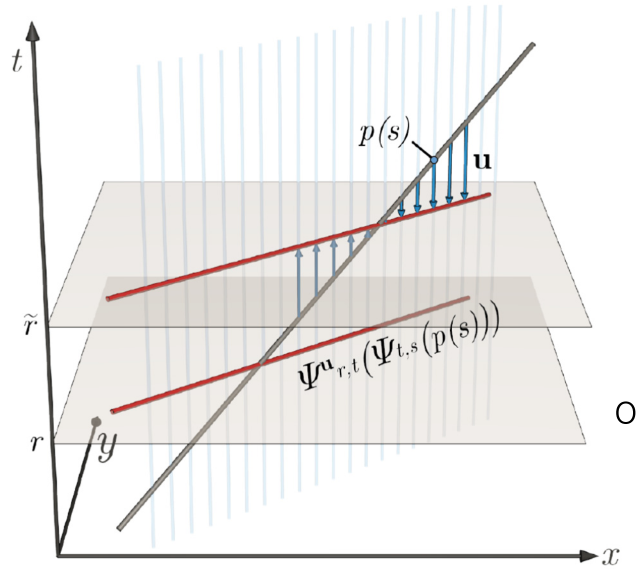
vorticity

Visualization depends on chosen **observation time**

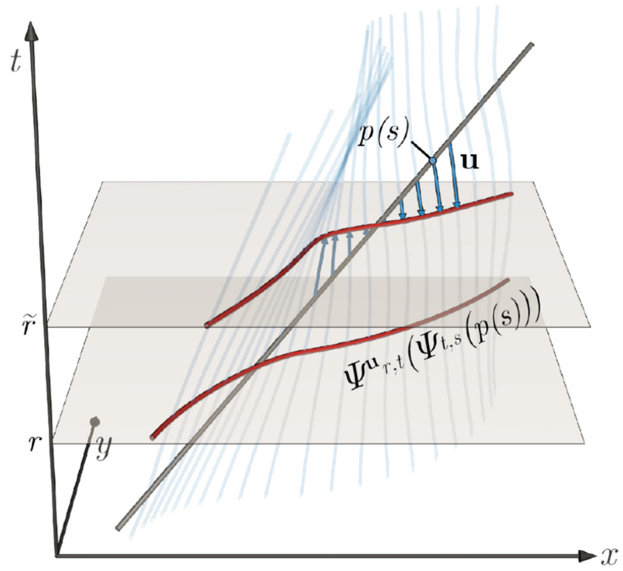
Visualization Using Observer Fields

Generalize stream/path/streak/time lines to **observed** variants

Can transform the input field according to observer field



different
observation times



Conclusions

One observer velocity field: Infinite set of reference frames

Observed time derivative (Lie derivative)

Global optimization: Approximately Killing, as steady as possible

Observer field-relative visualization

- Must choose **observation time**
- Can compute observation time field

Observed flow field is **objective**

- Objective flow features (direct computation or in observation time field)

Thank You!

Visit us at **`vccvisualization.org`**

MATLAB code on github!